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Module Categories and Modular Invariants

Hardiman, Leonard

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Leonard Hardiman

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

. . . .

Leonard Hardiman

Declaration of Authorship

I am the author of this thesis, and the work described therein was carried out by myself personally.

. . . .

Leonard Hardiman

«I’ve been cordially invited to join the visceral realists. I accepted, of course. There was no initiation ceremony. It was better that way. »

— Juan García Madero, *The Savage Detectives* by Roberto Bolaño

Abstract

Let $\mathcal{C}$ be a modular tensor category with a complete set of simples indexed by $\mathcal{I}$. A modular invariant for $\mathcal{C}$ is a non-negative integer $\mathcal{I} \times \mathcal{I}$-matrix that commutes with the modular data of $\mathcal{C}$. In this thesis we present a novel method of associating a non-negative integer $\mathcal{I} \times \mathcal{I}$-matrix to a pivotal monoidal functor $\mathcal{M}$ on $\mathcal{C}$. This is accomplished via a construction called the tube category. The tube category shares all of its objects with $\mathcal{C}$ but extends the Hom-spaces. The trace of $\mathcal{M}$ naturally extends to a representation of the tube category that we denote $\mathcal{TM}$. As irreducible representations of the tube category are indexed by pairs of elements in $\mathcal{I}$, decomposing $\mathcal{TM}$ into irreducibles gives a non-negative integer $\mathcal{I} \times \mathcal{I}$-matrix, $Z(\mathcal{TM})$. For a general pivotal functor, $Z(\mathcal{TM})$ will not always be a modular invariant; however, it will always commute with the T-matrix. Furthermore, under certain additional conditions on $\mathcal{M}$, it is shown that $\mathcal{TM}$ is a haploid, symmetric, commutative Frobenius algebra. Such algebras are known to be connected to modular invariants, in particular a result of Kong and Runkel implies that $Z(\mathcal{TM})$ commutes with the S-matrix if and only if the dimension of $\mathcal{TM}$ is equal to the dimension of $\mathcal{C}$. Finally, this procedure is applied to certain pivotal monoidal functors arising from module categories over $\mathcal{C}$.

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For Ella I offer no thanks as this thesis is hers as much as it is mine.

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TABLE OF NOTATION

$\mathbb{K}$	A field
$\underline{\text{Vect}}$	Category of finite dimensional $\mathbb{K}$-vector spaces
$¥$	Yoneda embedding
$X^\sharp$	Image of X under $¥ \colon \mathcal{C} \to \mathcal{RC}$
$\mathbf{1}$	Tensor Identity
$a_{X,Y,Z}$	Associativity isomorphism
r_X	Right unit isomorphism
l_X	Left unit isomorphism
an_X	Annihilation morphism
cr_X	Creation morphism
$\mathrm{Mod}\text{-}A$	Category of left A modules
$A, A\text{-}\mathrm{Bimod}$	Category of A-A bimodules
$\mathcal{RC}$	Category of contravariant representations of $\mathcal{C}$
$\mathrm{Irr}(\mathcal{C})$	Complete set of simple objects in $\mathcal{C}$
$\mathcal{TC}$	Tube category of $\mathcal{C}$
$\mathcal{RTC}$	Category of contravariant representations of $\mathcal{TC}$
$\mathcal{TM}$	The trace of $\mathcal{M}$ extended to a representation of $\mathcal{TC}$
$\hom_{\mathcal{C}}(X, Y)$	The dimension of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$
$\bar{F}$	The restriction of a functor on $\mathcal{TC}$ to $\mathcal{C}$
$\boxtimes$	Deligne tensor product
Tr	The trace functor $F \mapsto \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, F(-))$

All graphical descriptions of morphisms in monoidal categories are read vertically, top to bottom. For V and W vector spaces, we write "$V = W$" to indicate that V and W are isomorphic under an isomorphism that should be clear from the context.

CHAPTER 1

INTRODUCTION

1.1 | Background

In 1985 Segal wrote a landmark paper on a novel approach to the mathematical formulation of conformal field theory [Seg88]. He described a category $\underline{\mathrm{Seg}}$ whose objects are disjoint unions of a finite number of circles and whose morphisms are conformal equivalence classes of Riemann surfaces with parametrized boundary composed of incoming and outgoing circles. A (two dimensional) *conformal field theory* (CFT) is then a projective representation of this category satisfying certain axioms. This work proved to be hugely influential and has given rise to *functorial quantum field theory*, a broader attempt to provide a precise mathematical formulation of quantum field theory.

Let $\underline{\mathrm{Seg}}_0$ be the subcategory of $\underline{\mathrm{Seg}}$ whose morphisms have genus zero. Up to conformal equivalence, any morphism in $\underline{\mathrm{Seg}}_0$ is given by the complex unit disc with a finite number of non-intersecting sub-discs removed. A *vertex operator algebra* (VOA) may be thought of as a representation of $\underline{\mathrm{Seg}}_0$ that satisfies certain axioms including a holomorphic dependence on the centre and complex radius of the sub-discs[†]. An important property of a CFT is that it has two chiral halves: a holomorphic (or "left-moving") half and an anti-holomorphic (or "right-moving") half. In other words, the state space H of the theory decomposes into the direct

[†]This geometric approach to vertex operator algebras was developed by Huang (see, for example, [Hua97]).

sum

$$H = \bigoplus_{I,J} Z_{IJ}\; H_I \otimes \overline{H}_J \tag{1.1}$$

where the Z_{IJ} are multiplicity spaces and the H_I range over the irreducible modules of a VOA $\mathcal{V}$ (in this thesis we assume our CTFs are non-heterotic, i.e. that H_I and H_J are modules over the same VOA). The physical "uniqueness of the vacuum" assumption imposes that $Z_{\mathbf{1},\mathbf{1}} = \mathbb{C}$ where $\mathbf{1}$ is such that $H_{\mathbf{1}} = \mathcal{V}$. The CFT is called *rational* if $\mathcal{V}$ admits only finitely many irreducible modules; we assume that this is the case from now on.

Let $\mathbb{H}$ be the complex upper half plane, let τ be in $\mathbb{H}$ and let T_τ be the torus obtained by gluing together the annulus $\{z \in \mathbb{C} \mid |e^{2\pi i\tau}| < |z| < 1\}$. Let $Z\colon \mathbb{H} \to \mathbb{C}$ be the map which sends τ to the scalar that the CFT associates to T_τ. The decomposition of H given by (1.1) implies that Z may be written as the sum

$$Z(\tau) = \sum_{I,J} \dim Z_{IJ}\; \chi_I(\tau)\overline{\chi_J(\tau)} \tag{1.2}$$

where χ_I is the character of the irreducible VOA module H_I. This map is called the *partition function* of the theory. As a CFT should be invariant under conformal transformation and conformal structures on a torus are parametrized by $\mathbb{H}/\operatorname{PSL}_2(\mathbb{Z})$, we require that Z be invariant under the action of $\operatorname{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$; in other words, it should satisfy

$$Z(\tau+1) = Z(\tau) \quad \text{and} \quad Z\left(-\frac{1}{\tau}\right) = Z(\tau). \tag{1.3}$$

One of the most fascinating features of VOAs is their representation theory. The category of modules over a VOA has an extremely rich structure: it forms a *modular tensor category* (MTC). This surprising fact was first proved by Huang [Hua05]. MTCs possess many nice properties (they are semisimple, rigid, braided...) and in particular they come equipped with a representation of $\operatorname{PSL}_2(\mathbb{Z})$ given by their *modular data* (see Section 2.2.3). Let $\mathcal{I}$ be (an indexing set for) a complete set of irreducible objects in an MTC. The modular data is composed of two $\mathcal{I} \times \mathcal{I}$-matrices known as the S-matrix and the T-matrix; they are denoted by $\mathcal{S}$ and $\mathcal{T}$ respectively. The condition given by (1.3) may be rephrased as requiring that the $\mathcal{I} \times \mathcal{I}$-matrix with entries $\dim Z_{IJ}$ commutes with the modular data of the category of modules over the relevant VOA. This motivates the following definition.

DEFINITION 1.1.1. For an MTC with tensor identity $\mathbf{1}$, a *modular invariant* is a non-negative integer $\mathcal{I} \times \mathcal{I}$-matrix that commutes with the modular data and whose $(\mathbf{1}, \mathbf{1})$-entry is 1.

The VOA and partition function Z associated to a CFT capture much of the full theory. Indeed, knowing the partition function gives us the multiplicities $\dim Z_{IJ}$ and we may recover the space of states (up to isomorphism) via (1.1). A popular strategy when attempting to classify CFTs is to fix a VOA $\mathcal{V}$ and search for all compatible partition functions. From the above discussion we see that this is related to finding the modular invariants associated to the MTC of modules over $\mathcal{V}$. An example where this has been successfully carried out is provided by the VOA constructed from the affine Lie algebra $A_1^{(1)}$ together with a positive integer k, via the Sugawara construction [Sug68]. The category of modules in this case is the category of integrable highest weight modules of $A_1^{(1)}$ at level k, denoted $\mathrm{Rep}_k\, A_1^{(1)}$. The modular data of this category is given by the Kac-Peterson matrices whose entries (up to a scalar) are

$$\mathcal{S}_{ab} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{ab}{k+2}\right) \quad \text{and} \quad \mathcal{T}_{ab} = \exp\left(\pi i \frac{a^2}{2(k+2)}\right) \delta_{a,b}.$$

In 1986 Cappelli, Itzykson and Zuber classified all possible modular invariants in this context and, to their surprise, the classification followed an A-D-E pattern [CIZ87]. The appearance of this pattern intrigued many researchers in the field and was the subject of much speculation [Gan00a, Zub02, KO02]. The first explanation of the pattern was provided by an operator algebra technique know as *α-induction*, due to Böckenhauer and Evans [BE98]. This technique relates the A-D-E classification of Goodman-de la Harpe-Jones subfactors to Cappelli, Itzykson and Zuber's classification [BE01, Ocn99].

When translating from the operator algebra language to the purely categorical one an inclusion of subfactors corresponds to a module category[†]. It was sub-

[†]Just as a monoidal category may be thought of as the categorification of a ring, a module category may be thought of as the categorification of a module.

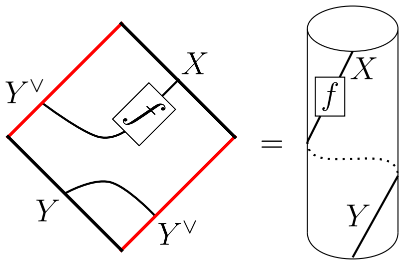
sequently shown that the classification of finite *module categories* over $\operatorname{Rep}_k A_1^{(1)}$ also follows an A-D-E pattern [EO04]. Further understanding the relationship between module categories and modular invariants was one of the motivating goals throughout this thesis.

Finite modules categories also have a physical interpretation. In 1989 Cardy showed that the algebraic data of an *annular partition function* in a *boundary* CFT (as opposed to the toroidal partition function Z) is given by a finite module category over the corresponding MTC [Car89]. From a physical point of view the correspondence between module categories and modular invariants should therefore be thought of as a "closing up" just as an annulus closes up into a torus. Mathematically we would expect this "closing up" to correspond to taking the *trace*, in some suitable sense, of the module category. A notion of trace does exists for module categories (and more generally for monoidal functors), however it simply produces a representation of the MTC. A priori it is not at all clear how to associate a non-negative integer $\mathcal{I} \times \mathcal{I}$-matrix to this representation. This thesis presents a solution to this problem by extending the representation to take values on the *tube category* of the underlying MTC.

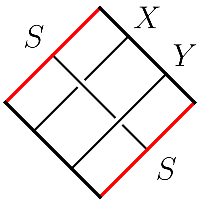
1.2 | Content of this Thesis

Chapter 2 starts by developing the background material on (pre)-modular tensor categories and describing the graphical notation we will use to work with them. While Section 2.2.3 contains the definitions of the main characters in this thesis, i.e. modular tensor categories and their modular data, the definition of the pivotal structure in Section 2.2.2 also plays a significant role. The chapter then moves on to Section 2.3 which proves some preliminary results within this context; many of which have previously appeared in [HK19]. The most important results from this section are Lemma 2.3.8 and Proposition 2.3.13 as they will both be used multiple times throughout this thesis.

Chapter 3 then introduces the tube category, denoted $\mathcal{TC}$, of a spherical fusion category $\mathcal{C}$. This category shares all of its objects with $\mathcal{C}$ but extends the Hom-spaces. In particular, morphisms in $\mathcal{TC}$ may be thought of as morphisms in $\mathcal{C}$ drawn on a *cylinder*. For example, for every morphism $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$, there is a morphism

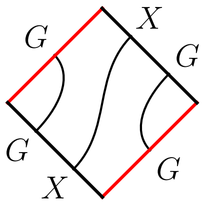


in $\mathrm{Hom}_{\mathcal{TC}}(X, Y)$. When $\mathcal{C}$ is pre-modular our principal tools for studying $\mathcal{TC}$ are the following idempotents,

$$\epsilon_X^Y = \frac{1}{d(\mathcal{C})} \bigoplus_S d(S)$$  $$\in \mathrm{End}_{\mathcal{TC}}(X \otimes Y).$$

In the case when $\mathcal{C}$ is modular, Theorem 3.1.13 computes the Hom-spaces between these idempotents. This immediately implies Corollary 3.1.14 which states that the set $\{\epsilon_I^J\}_{I,J \in \mathrm{Irr}(\mathcal{C})}$ is a set of orthogonal primitive idempotents, where $\mathrm{Irr}(\mathcal{C})$ is a complete set of simples in $\mathcal{C}$. Theorem 3.1.15 then goes on to prove that $\{\epsilon_I^J\}_{I,J \in \mathrm{Irr}(\mathcal{C})}$ is a *complete* set of orthogonal primitive idempotents, i.e. that every morphism in the tube category factors through the ϵ_I^J. Let $\mathcal{RTC}$ denote the category of representations of $\mathcal{TC}$. Theorem 3.1.15 implies Corollary 3.1.16 which states that applying the Yoneda embedding to the ϵ_I^J idempotents gives a complete set of simples in $\mathcal{RTC}$. In particular, simple objects in $\mathcal{RTC}$ are indexed by pairs of elements in $\mathrm{Irr}(\mathcal{C})$. Many of these results have previously appeared, with different proofs, in [HK19].

Returning to the more general case of when $\mathcal{C}$ is a spherical fusion category, Proposition 3.2.1 characterises the data required to extend a representation of $\mathcal{C}$ to a representation of $\mathcal{TC}$. In particular it shows that specifying the value of the representation on morphisms of the form

$c_{G,X} =$ 

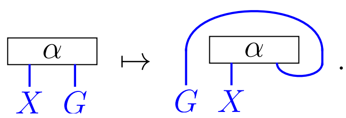
entirely determines the representation on $\mathcal{TC}$. This result, together with the content of Section 3.3.1, provides an alternative proof of the equivalence between the category of representations of $\mathcal{TC}$ and the *centre* of $\mathcal{C}$ (this equivalence was already known [PSV18, Proposition 3.14]). The centre of $\mathcal{C}$, denoted $Z(\mathcal{C})$, is a category whose objects consist of an object in $\mathcal{C}$ together with a half braiding. When $\mathcal{C}$ is braided we get a canonical choice of half braiding for ever object in $\mathcal{C}$, this gives us a braided monoidal functor $\mathcal{C} \to Z(\mathcal{C})$. The same holds for $\overline{\mathcal{C}}$ (the category obtained by equipping $\mathcal{C}$ with the opposite braiding) and we obtain a braided monoidal functor

$$\Phi\colon\ \mathcal{C} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{C}) \xrightarrow{\cong} \mathcal{RTC}.$$

Theorem 3.3.3 then shows that Φ maps $X \boxtimes Y$ to the Yoneda embedding of the idempotent ϵ_X^Y. Together with previous results this yields an alternative proof that Φ is an equivalence when $\mathcal{C}$ is modular. This was first proved in [Müg03].

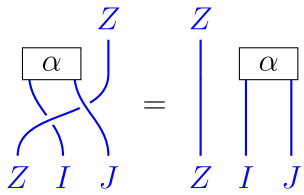
In Chapter 4, we associate a representation of the tube category to a *pivotal* monoidal functor $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$. This is accomplished by first considering the *trace* of $\mathcal{M}$, i.e. the representation of $\mathcal{C}$ given by $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(-))$. It is then shown that this representation of $\mathcal{C}$ naturally extends to a representation of the tube category. Indeed, the value of this extension on $c_{G,X}$ is given by

$$\mathrm{Tr}\,\mathcal{M}(XG) \to \mathrm{Tr}\,\mathcal{M}(GX)$$



where the blue lines should be interpreted as strands in $\mathcal{C}$ that have been evaluated under $\mathcal{M}$. We denote this extension $\mathcal{TM}$. We now assume that $\mathcal{C}$ is modular. As simple objects in $\mathcal{RTC}$ are indexed by pairs of elements in $\mathrm{Irr}(\mathcal{C})$, decomposing

$\mathcal{TM}$ into simple summands gives a non-negative integer $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$-matrix that we denote $Z(\mathcal{TM})$. Proposition 4.1.1 then provides a graphical description of the simple multiplicity spaces in $\mathcal{TM}$ as the subspace of $\mathcal{TM}(IJ)$ defined by the conditions that $\alpha \in \mathcal{TM}(IJ)$ satisfy



(1.4)

for all Z in $\mathcal{C}$.

We call a general representation of the tube category *T-invariant* if its decomposition into simple summands commutes with the T-matrix of $\mathcal{C}$. Theorem 4.2.4 gives a graphical characterisation of T-invariance. An immediate corollary of this is that $\mathcal{TM}$ is T-invariant. This corollary will later be strengthened to Theorem 4.4.7 which does not assume that $\mathcal{C}$ is modular.

We call a general representation of the tube category *S-invariant* if its decomposition into simple summands commutes with the S-matrix of $\mathcal{C}$. In general, $Z(\mathcal{TM})$ fails to be S-invariant. Indeed, when $\mathcal{M} = \mathrm{id}_{\mathcal{C}}$, $Z(\mathcal{TM})$ is given by

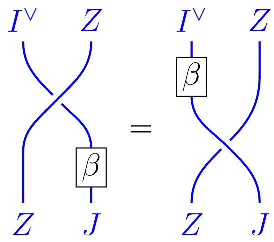
$$Z(\mathcal{TM})_{IJ} = \begin{cases} 1 & \text{if } I = J = \mathbf{1} \\ 0 & \text{else} \end{cases}$$

which doesn't commute with the S-matrix in general (this is explained in greater detail in Example 4.3.2). However, under the assumption that $\mathcal{M}$ is indecomposable and takes value in a category whose idempotent completion is multifusion, Theorem 4.3.23 proves that $\mathcal{TM}$ is a haploid, symmetric, commutative, Frobenius algebra. By a result of Kong and Runkel [KR09, Theorem 3.4] this implies that $Z(\mathcal{TM})$ commutes with the S-matrix if and only if the dimension of $\mathcal{TM}$ is equal to the dimension of $\mathcal{C}$. This condition on the dimension of $\mathcal{TM}$ is equivalent to requiring that

$$(\mathcal{S}\ Z(\mathcal{TM})\ \mathcal{S}^{-1})_{\mathbf{1},\mathbf{1}} = Z(\mathcal{TM})_{\mathbf{1},\mathbf{1}}$$

and is therefore always a necessary condition for S-invariance.

The final section in Chapter 4 discusses the operator algebra technique known as α-induction. In particular it describes a categorical formulation of α-induction given by Ostrik [Ost03, Section 5]. A module category may be though of as a (not necessarily pivotal) monoidal functor $\mathcal{M}\colon \mathcal{C} \to A, A\text{-Bimod}$, where A is a semisimple algebra. Ostrik states that the integer, non-negative $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$-matrix produced by α-induction has entries given by the dimension of the subspace of $\mathrm{Hom}_{A,A\text{-Bimod}}(I^\vee, J)$ defined by the relations



(1.5)

for all Z in $\mathcal{C}$. Theorem 4.4.6 proves that, when $\mathcal{M}$ *induces a pivotal structure on its image*, the $\mathcal{TM}$ construction may be applied and $Z(\mathcal{TM})$ will produce the same matrix as α-induction. This is accomplished by relating Condition (1.4) and Condition (1.5). Furthermore, this application of the $\mathcal{TM}$ construction to module categories leads us to Corollary 4.4.8 which states that, when $\mathcal{M}$ is an indecomposable module category that induces a pivotal structure on its full image, $\mathcal{TM}$ is a haploid, symmetric, commutative, Frobenius algebra.

Finally, Chapter 5 applies the theory developed throughout the thesis to a class of examples arising from module categories over the Temperley-Lieb category. The Temperley-Lieb category may be thought of as a diagrammatic presentation of the previously discussed category of integrable, highest weight modules of $A_1^{(1)}$ at level k. Indeed, as described in Section 5.1.4, the Temperley-Lieb category possesses a unique proper tensor ideal; after quotienting by this ideal the category admits the structure of an MTC which is equivalent to that of $\mathrm{Rep}_k\, A_1^{(1)}$. A recipe for producing module categories over the Temperley-Lieb category is then described, the input of which is a symmetric quiver satisfying certain conditions. Proposition 5.2.2 then proves that the module categories this recipe produces induce a pivotal structure on their full image. The A-D-E classification of module categories over $\mathrm{Rep}_k\, A_1^{(1)}$ [EO04] then implies that all module categories over $\mathrm{Rep}_k\, A_1^{(1)}$ arise from this recipe. The chapter culminates in Section 5.2.3 where the $\mathcal{TM}$ construction is used to give a new explanation of the A-D-E pattern that appears in the Cappelli-Itzykson-Zuber classification of $A_1^{(1)}$ modular invariants.

CHAPTER 2 MODULAR TENSOR CATEGORIES

As mentioned in the introduction, the category of modules over a vertex operator algebra forms a *modular tensor category*. The aim of this chapter is to give an exposition of such categories and to prove certain graphical formulas that will be exploited in Chapter 3.

Section 2.1 starts by introducing linear categories, the direct sum construction and the product of an object with a vector space. The Yoneda Lemma is then exploited to provide a universal construction of the idempotent completion of any linear category.

Section 2.2 introduces the notion of a monoidal category and the corresponding notion of duals. It then goes on to discuss pivotal structures and braidings. Finally, the definition of a modular tensor category is given. Throughout this section a graphical notation is introduced and developed.

Section 2.3 starts by giving a graphical description of how objects may be decomposed into their simple summands for a relatively general class of linear category. The context is then restricted to modular tensor categories and certain consequences of the *killing ring lemma* [BK01, Corollary 3.1.11.] are developed. The results in this section are well-known to experts in the field, however the proofs are original. Much of the material in this section has previously appeared in [HK19].

2.1 | Preliminary Category Theory

2.1.1 | Linear Categories

Let $\mathbb{K}$ be a field. In this thesis all categories are *linear categories over* $\mathbb{K}$, i.e. the Hom-spaces are finite dimensional vector spaces over $\mathbb{K}$ and composition is bilinear.

EXAMPLE 2.1.1. The category of finite dimensional vector spaces $\underline{\mathrm{Vect}}$ is a linear category as $\mathrm{Hom}(V, W)$ is a $\dim(V) \times \dim(W)$-dimensional vector space.

Furthermore all functors are *linear functors*, i.e. functors between two linear categories such that the corresponding maps between Hom-spaces are linear.

DEFINITION 2.1.2. Let X and Y be objects in a category $\mathcal{C}$. A *direct sum of X and Y* is an object Z in $\mathcal{C}$ such that

- $\mathrm{Hom}_{\mathcal{C}}(Z, A)$ is naturally identified with $\mathrm{Hom}_{\mathcal{C}}(X, A) \oplus \mathrm{Hom}_{\mathcal{C}}(Y, A)$.
- $\mathrm{Hom}_{\mathcal{C}}(A, Z)$ is naturally identified with $\mathrm{Hom}_{\mathcal{C}}(A, X) \oplus \mathrm{Hom}_{\mathcal{C}}(A, Y)$.

DEFINITION 2.1.3. Let X be in $\mathcal{C}$ and let V be in $\underline{\mathrm{Vect}}$. The *product of V with X* is an object Z in $\mathcal{C}$ such that

- $\mathrm{Hom}_{\mathcal{C}}(Z, Y)$ is naturally identified with $V^* \otimes \mathrm{Hom}_{\mathcal{C}}(X, Y)$.
- $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ is naturally identified with $V \otimes \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

REMARK 2.1.4. For fixed X, Y in $\mathcal{C}$ and V in $\underline{\mathrm{Vect}}$, one can check that both the direct sum of X and Y and the product of V with X are unique. This can also be seen as a consequence of the Yoneda Lemma. They are denoted $X \oplus Y$ and $V \cdot X$ respectively.

In practice we never consider the question of whether or not direct sums or products exist. This is due to the fact that, if they do not exist, they may be formally added unambiguously. The following lemma captures the relationship between products and direct sums.

LEMMA 2.1.5. *Let be V in $\underline{\mathrm{Vect}}$, let b be a basis of V and let X be in $\mathcal{C}$. Then*

$$V \cdot X = \bigoplus_b X.$$

Proof. Let b^* be the dual basis to b. The maps $\bigoplus_b b \otimes \mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_b X, V \cdot X\right)$ and $\bigoplus_b b^* \otimes \mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}\left(V \cdot X, \bigoplus_b X\right)$ are inverse to one another.

□

We recall that an object X in $\mathcal{C}$ is called *simple* if X has no proper subobjects. Schur's Lemma implies that, for a simple object S in $\mathcal{C}$, $\mathrm{End}_{\mathcal{C}}(S)$ is a division algebra over $\mathbb{K}$. We call an object X *Schurian* if $\mathrm{End}_{\mathcal{C}}(X) = \mathbb{K}$. We call a category Schurian if all of its simple objects are Schurian. In particular if $\mathbb{K}$ is algebraically closed then $\mathcal{C}$ is Schurian.

DEFINITION 2.1.6. For a category $\mathcal{C}$ a *complete set of simples* $\mathrm{Irr}(\mathcal{C})$ is a set such that

- □ for all I in $\mathrm{Irr}(\mathcal{C})$, I is a simple object in $\mathcal{C}$.
- □ for all simple object S in $\mathcal{C}$ there exists a unique $I \in \mathrm{Irr}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{C}}(S, I) \neq 0$.

We recall that a category $\mathcal{C}$ is called *semisimple* if every object in $\mathcal{C}$ is a direct sum of finitely many simple objects.

DEFINITION 2.1.7. A *finite* category is a semisimple category that admits a finite complete set of simples.

The following canonical decomposition of an object in a semisimple category will be of great importance for the remainder of this thesis.

PROPOSITION 2.1.8. *Let $\mathcal{C}$ be a semisimple, Schurian category, let $\mathrm{Irr}(\mathcal{C})$ be a complete set of simples and let X be an object in $\mathcal{C}$. Then*

$$X = \bigoplus_{S \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(S, X) \cdot S.$$

Proof. As X is semisimple we have

$$X = \bigoplus_{i \in \mathcal{I}} X_i$$

where the X_i are simple objects and $\mathcal{I}$ is an indexing set. We therefore have

$$\bigoplus_{S \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(S, X) \cdot S = \bigoplus_{\substack{i \in \mathcal{I} \\ S \in \mathrm{Irr}(C)}} \mathrm{Hom}_{\mathcal{C}}(S, X_i) \cdot S = \bigoplus_{\substack{i \in \mathcal{I}_S \\ S \in \mathrm{Irr}(C)}} \mathrm{Hom}_{\mathcal{C}}(S, X_i) \cdot S$$

where $\mathcal{I}_S = \{i \in \mathcal{I} \mid X_i \cong S\}$. As $\mathcal{C}$ is Schurian $\mathrm{Hom}_{\mathcal{C}}(S, X_i)$ is one-dimensional and the canonical morphism

$$\begin{aligned} \mathrm{id} \in \mathrm{End}(\mathrm{Hom}_{\mathcal{C}}(S, X_i)) &= \mathrm{Hom}_{\mathcal{C}}(S, X_i)^* \otimes \mathrm{Hom}_{\mathcal{C}}(S, X_i) \\ &= \mathrm{Hom}_{\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}}(S, X_i) \cdot S, X_i) \end{aligned}$$

is an isomorphism by Schur's Lemma. Therefore

$$\bigoplus_{\substack{i \in \mathcal{I}_S \\ S \in \mathrm{Irr}(C)}} \mathrm{Hom}_{\mathcal{C}}(S, X_i) \cdot S = \bigoplus_{\substack{i \in \mathcal{I}_S \\ S \in \mathrm{Irr}(C)}} X_i = X.$$

□

2.1.2 | IDEMPOTENT COMPLETION AND THE YONEDA EMBEDDING

DEFINITION 2.1.9. Let X be an object in a category. A morphism $\varepsilon \in \mathrm{End}_{\mathcal{C}}(X)$ such that $\varepsilon \circ \varepsilon = \varepsilon$ is called an idempotent.

Let V be in Vect and suppose $\varepsilon \in \mathrm{End}(V)$ is an idempotent. In this case $V_\varepsilon = \{v \in V \mid \varepsilon(v) = v\}$ forms a subspace of V and ε may be thought of as a projection from V onto V_ε composed with an inclusion of V_ε back into V. Furthermore this correspondence between idempotents on V and split subspaces of V defines a bijection. To say this more generally we made the following definition,

DEFINITION 2.1.10. Let $\varepsilon \in \operatorname{End}_{\mathcal{C}}(X)$ be an idempotent in a category $\mathcal{C}$. An *image object* for ε is a object X_ε in $\mathcal{C}$ together with morphisms $\pi\colon X \to X_\varepsilon$ and $i\colon X_\varepsilon \to X$ such that $i \circ \pi = \varepsilon$ and $\pi \circ i = \mathrm{id}_{X_\varepsilon}$.

REMARK 2.1.11. Suppose (X_1, π_1, i_1) and (X_2, π_2, i_2) are two image objects for an idempotent $\varepsilon \in \operatorname{End}_{\mathcal{C}}(X)$. Then $\pi_2 \circ i_1$ and $\pi_1 \circ i_2$ give inverse isomorphisms between X_1 and X_2. Furthermore, as the diagram

$$\begin{tikzcd}
& X \arrow[dl, shift left, "\pi_1"] \arrow[dr, shift right, "\pi_2"'] & \\
X_1 \arrow[ur, shift left, "i_1"] \arrow[rr, shift left, "\pi_2 \circ i_1"] & & X_2 \arrow[ul, shift right, "i_2"'] \arrow[ll, shift left, "\pi_1 \circ i_2"]
\end{tikzcd}$$

commutes, image objects of ε are unique as summands of X.

Now let $\mathcal{C}$ be any (linear) category. If there exists an image object for every idempotent in $\mathcal{C}$ we say that $\mathcal{C}$ is *idempotent complete*. This property fails for many categories (an example can be found in Section 5.1.3). In this case it may be desirable to fully embed $\mathcal{C}$ into another category $\overline{\mathcal{C}}$ that is idempotent complete.

DEFINITION 2.1.12. An *idempotent completion* of a category $\mathcal{C}$ is a category $\overline{\mathcal{C}}$ together with a covariant functor $\Phi\colon \mathcal{C} \to \overline{\mathcal{C}}$ such that

- □ Φ is fully faithful.
- □ $\overline{\mathcal{C}}$ is idempotent complete.
- □ For every object X in $\overline{\mathcal{C}}$ there exists an idempotent ε in $\mathcal{C}$ such that X is an image object for $\Phi(\varepsilon)$.

REMARK 2.1.13. Idempotent completions are unique up to equivalence of categories [Lur09, Section 5.1.4].

We shall now describe a realization of an idempotent completion for any category $\mathcal{C}$. Let $\mathcal{RC}$ denote the category of *contravariant* functors from $\mathcal{C}$ into $\underline{\text{Vect}}$.

We consider the functor

$$\begin{aligned} ¥ \colon \mathcal{C} &\to \mathcal{RC} \\ X &\mapsto X^\sharp \end{aligned}$$

where $X^\sharp = \mathrm{Hom}_\mathcal{C}(-, X)$.

It is a well known corollary of the Yoneda Lemma that ¥ is fully faithful and is therefore referred to as the *Yoneda embedding*. Furthermore for any idempotent $\varepsilon \in \mathrm{End}_\mathcal{C}(X)$, there is, in $\mathcal{RC}$, a subfunctor $(X, \varepsilon)^\sharp \leq X^\sharp$ given by

$$\begin{aligned} (X, \varepsilon)^\sharp \colon \mathcal{C} &\to \underline{\mathrm{Vect}} \\ Y &\mapsto \mathrm{Hom}_\mathcal{C}(Y, \varepsilon) := \{ f \in \mathrm{Hom}_\mathcal{C}(Y, X) \mid \varepsilon \circ f = f \} \\ (\alpha \colon Y \to Z) &\mapsto (f \mapsto \alpha \circ f) \end{aligned}$$

which is an image object for $¥(\varepsilon)$ in $\mathcal{RC}$. Indeed, $(X, \varepsilon)^\sharp$ is a summand of $X^\sharp$. This image object exists because $\mathcal{RC}$ is an abelian category, so idempotent complete, even if $\mathcal{C}$ may not be. Concretely, $(X, \varepsilon)^\sharp(Y)$ is the image of $\varepsilon^\sharp_Y = \varepsilon_*$, which is an idempotent endomorphism of $X^\sharp(Y) = \mathrm{Hom}_\mathcal{C}(Y, X)$. The naturality of $\varepsilon^\sharp$, i.e. the fact that ε_* commutes with ϕ^* for any $\phi \colon Z \to Y$, makes $(X, \varepsilon)^\sharp$ a functor.

Let $\overline{\mathcal{C}}_¥$ be the full subcategory of $\mathcal{RC}$ spanned by object of the form $(X, \varepsilon)^\sharp$. We note that ¥ factors through $\overline{\mathcal{C}}_¥$ as $(X, \mathrm{id}_X)^\sharp = ¥(X)$ for all X in $\mathcal{C}$. We therefore obtain the following result.

PROPOSITION 2.1.14. *The category $\overline{\mathcal{C}}_¥$ together with the Yoneda embedding is an idempotent completion of $\mathcal{C}$.*

PROPOSITION 2.1.15. *Let $\mathcal{C}$ be a finite Schurian category. Then the Yoneda embedding*

$$\begin{aligned} \mathcal{C} &\to \mathcal{RC} \\ X &\mapsto X^\sharp \end{aligned}$$

is an equivalence.

Proof. As the Yoneda embedding is fully faithful we only have to show that it is essentially surjective. For F in $\mathcal{RC}$ and X in $\mathcal{C}$, we have

$$\begin{aligned} F(X) &= \bigoplus_{S \in \mathrm{Irr}(\mathcal{C})} F(S) \otimes \mathrm{Hom}_C(S, X)^* \\ &= \bigoplus_{S \in \mathrm{Irr}(\mathcal{C})} F(S) \otimes \mathrm{Hom}_C(X, S) \\ &= \bigoplus_{S \in \mathrm{Irr}(\mathcal{C})} F(S) \otimes S^\sharp(X) \end{aligned}$$

where the first equality uses the semisimplicity of $\mathcal{C}$ and the contravariance of F and the second equality uses the fact S is Schurian.

□

Let $\mathcal{C}$ be a semisimple category together with a complete set of simples $\mathrm{Irr}(\mathcal{C})$. Suppose we choose an element $I \in \mathrm{Irr}(\mathcal{C})$ and consider the full subcategory of $\mathcal{C}$ whose objects are non-isomorphic to I. Clearly this new category fails to be semisimple, however the missing simple objects may still be detected by considering the idempotent endomorphisms of any object that has a proper summand isomorphic to I. There is, therefore, a notion analogous to a complete set of simples for idempotents.

DEFINITION 2.1.16. A *set of primitive orthogonal idempotents* in a linear category $\mathcal{C}$ is a set of idempotents $\mathcal{I}$ in $\mathcal{C}$ such that

$$\mathrm{Hom}_{\mathcal{C}}(\varepsilon, \varepsilon') = \begin{cases} \mathbb{K} & \text{if } \varepsilon = \varepsilon' \\ 0 & \text{else.} \end{cases}$$

A set of primitive orthogonal idempotents is called *complete* if we have

$$\bigoplus_{\varepsilon \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X, \varepsilon) \otimes \mathrm{Hom}_{\mathcal{C}}(\varepsilon, Y) = \mathrm{Hom}_C(X, Y)$$

for all X, Y in $\mathcal{C}$.

The proof Proposition 2.1.15 shows that the Yoneda embedding maps a complete set of Schurian simples in $\mathcal{C}$ to a complete set of Schurian simples in $\mathcal{RC}$. The corresponding claim for a complete set of primitive orthogonal idempotents also holds.

PROPOSITION 2.1.17. *Let $\mathcal{C}$ be a linear category with a complete set of primitive orthogonal idempotents $\mathcal{I}$. Then $\mathcal{RC}$ is a semisimple Schurian category and $\{(X_\varepsilon, \varepsilon)^\sharp\}_{\varepsilon\in\mathcal{I}}$ forms a complete set of simples in $\mathcal{RC}$ (where $\varepsilon \in \mathrm{End}_{\mathcal{C}}(X_\varepsilon)$).*

Proof. It is simple to check that the set $\{(X_\varepsilon, \varepsilon)^\sharp\}_{\varepsilon\in\mathcal{I}}$ contains distinct simple Schurian objects in $\mathcal{RC}$. The condition that $\mathcal{I}$ is a complete set of orthogonal idempotents may be rephrased as

$$Y^\sharp = \bigoplus_{\varepsilon\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\varepsilon, Y) \cdot (X_\varepsilon, \varepsilon)^\sharp$$

for all Y in $\mathcal{C}$. Then, using a similar argument to the proof of Proposition 2.1.15, we take F in $\mathcal{RC}$, Y in $\mathcal{C}$ and compute,

$$\begin{aligned} F(Y) &= \mathrm{Hom}_{\mathcal{RC}}(Y^\sharp, F) \\ &= \bigoplus_{\varepsilon\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(\varepsilon, Y)^* \otimes \mathrm{Hom}_{\mathcal{RC}}((X_\varepsilon, \varepsilon)^\sharp, F) \\ &= \bigoplus_{\varepsilon\in\mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(Y, \varepsilon) \otimes \mathrm{Hom}_{\mathcal{RC}}((X_\varepsilon, \varepsilon)^\sharp, F) \\ &= \bigoplus_{\varepsilon\in\mathcal{I}} \mathrm{Hom}_{\mathcal{RC}}((X_\varepsilon, \varepsilon)^\sharp, F) \otimes (X_\varepsilon, \varepsilon)^\sharp(Y) \end{aligned}$$

as desired.

□

COROLLARY 2.1.18. *Let $\mathcal{C}$ be a linear category with a complete set of primitive orthogonal idempotents. Then $\mathcal{RC}$ together with the Yoneda embedding give an idempotent completion of $\mathcal{C}$.*

Proof. This follows from Proposition 2.1.14 and Proposition 2.1.17.

□

2.2 | MONOIDAL CATEGORIES

2.2.1 | INTRODUCTION TO MONOIDAL CATEGORIES

DEFINITION 2.2.1. A *monoidal category* is a category $\mathcal{C}$ together with a *tensor product* bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ with natural *associativity isomorphisms* $a_{X,Y,Z}\colon (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z)$ and a *tensor identity* $\mathbf{1}$ in $\mathcal{C}$ with natural *unit isomorphisms* $r_X\colon X\otimes\mathbf{1} \to X$ and $l_X\colon\ \mathbf{1}\otimes X \to X$ such that

□ The diagram

$$\begin{tikzcd}
& ((X\otimes Y)\otimes Z)\otimes W \arrow[dl, "a_{X,Y,Z}\otimes \mathrm{id}"'] \arrow[dr, "a_{X\otimes Y,Z,W}"] & \\
(X\otimes(Y\otimes Z))\otimes W \arrow[d, "a_{X,Y\otimes Z,W}"] & & (X\otimes Y)\otimes(Z\otimes W) \arrow[d, "a_{X,Y,Z\otimes W}"'] \\
X\otimes((Y\otimes Z)\otimes W) \arrow[rr, "\mathrm{id}\otimes a_{Y,Z,W}"] & & X\otimes(Y\otimes(Z\otimes W))
\end{tikzcd}$$

commutes.

□ The diagram

$$\begin{tikzcd}
(X\otimes\mathbf{1})\otimes Y \arrow[rr, "a_{X,\mathbf{1},Y}"] \arrow[dr, "r_X\otimes\mathrm{id}"'] & & X\otimes(\mathbf{1}\otimes Y) \arrow[dl, "\mathrm{id}\otimes l_Y"] \\
& X\otimes Y &
\end{tikzcd}$$

commutes.

REMARK 2.2.2. By interpreting objects as 1-morphisms and $\otimes$ as composition of 1-morphisms we see that the notion of a monoidal category is equivalent to the notion of a 2-category with exactly one object.

REMARK 2.2.3. For the remainder of this thesis we will suppress the associativity and unit isomorphisms.

DEFINITION 2.2.4. Let $\mathcal{C}$ be a monoidal category and let X be an object in $\mathcal{C}$. An object Y in $\mathcal{C}$ together with morphisms

$$\mathrm{cr}\colon\ \mathbf{1} \to X \otimes Y \quad \text{and} \quad \mathrm{an}\colon Y \otimes X \to \mathbf{1}$$

such that

$$(\mathrm{id}_X \otimes \mathrm{an}) \circ (\mathrm{cr} \otimes \mathrm{id}_X) = \mathrm{id}_X \tag{2.1}$$

and

$$(\mathrm{an} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_Y \otimes \mathrm{cr}) = \mathrm{id}_Y \tag{2.2}$$

is called a *right* dual to X. An object Z in $\mathcal{C}$ together with morphisms

$$\mathrm{cr}\colon\ \mathbf{1} \to Z \otimes X \quad \text{and} \quad \mathrm{an}\colon X \otimes Z \to \mathbf{1}$$

is called a *left* dual to X if X, together with the same morphisms, is a right dual to Z. The maps cr and an are called the *creation and annihilation morphisms* respectively.

REMARK 2.2.5. If X in $\mathcal{C}$ admits a right (resp. left) dual, then the dual is unique [EGNO15, Proposition 2.10.5.].

DEFINITION 2.2.6. A monoidal category $\mathcal{C}$ is called *rigid if every object in $\mathcal{C}$ admits a left and a right dual.*

For an object X in $\mathcal{C}$ we use $X^\vee$ to denote a right dual to X and ${}^\vee X$ to denote a left dual to X. The corresponding creation and annihilation morphisms are denoted cr_X and an_X respectively.

LEMMA 2.2.7.

$$(X \otimes Y)^\vee = Y^\vee \otimes X^\vee \quad \textit{and} \quad {}^\vee(X \otimes Y) = {}^\vee Y \otimes {}^\vee X.$$

Proof. The isomorphism

$$\begin{aligned}(X \otimes Y)^\vee &\xrightarrow{\mathrm{id} \otimes \mathrm{cr}_X} (X \otimes Y)^\vee \otimes X \otimes X^\vee \\ &\xrightarrow{\mathrm{id} \otimes \mathrm{cr}_Y \otimes \mathrm{id}} (X \otimes Y)^\vee \otimes X \otimes Y \otimes Y^\vee \otimes X^\vee \xrightarrow{\mathrm{an}_{X \otimes Y} \otimes \mathrm{id}} Y^\vee \otimes X^\vee\end{aligned}$$

gives the first identify, the second may be proved analogously.

□

LEMMA 2.2.8.

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) = \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^\vee), \quad \mathrm{Hom}_{\mathcal{C}}(X, Y \otimes Z) = \mathrm{Hom}_{\mathcal{C}}(Y^\vee \otimes X, Z)$$
$$\mathrm{Hom}_{\mathcal{C}}(X \otimes {}^\vee Y, Z) = \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y), \quad \mathrm{Hom}_{\mathcal{C}}(Y \otimes X, Z) = \mathrm{Hom}_{\mathcal{C}}(X, {}^\vee Y \otimes Z).$$

Proof. We consider the following canonical map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) &\to \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^\vee) \\ g &\mapsto (g \otimes \mathrm{id}_{Y^\vee}) \circ (\mathrm{id}_X \otimes \mathrm{cr}_Y) \end{aligned}$$

it has inverse

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^\vee) &\to \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z)) \\ g &\mapsto (\mathrm{id}_Z \otimes \mathrm{an}_Y) \circ (g \otimes \mathrm{id}_Y) \end{aligned}$$

and is therefore a canonical isomorphism. The proofs of the other equalities are analogous.

□

COROLLARY 2.2.9.

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y^\vee, X^\vee) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes X^\vee).$$

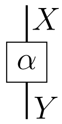
Let $\mathcal{C}$ be a rigid category. Lemma 2.2.5 and Corollary 2.2.9 give us contravariant endofunctors $-^\vee$ and ${}^\vee-$ on $\mathcal{C}$. The following rephrasing of Lemma 2.2.8 in terms of the Yoneda embedding will prove useful.

LEMMA 2.2.10. *Let $\mathcal{C}$ be a rigid monoidal category. Then we have*

$$Y^\sharp \circ (X^\vee \otimes -) = (X \otimes Y)^\sharp = X^\sharp \circ (- \otimes {}^\vee Y).$$

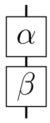
Proof. The natural isomorphisms from the proof of Lemma 2.2.8 give the desired result. □

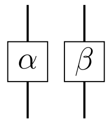
We now introduce a graphical notation, due to Roger Penrose [Pen71], that shall be used throughout the remainder of this thesis. To represent a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we draw a strand labelled X, a strand labelled Y and a connection between them labelled α as follows



.

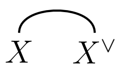
Composition is then depicted by vertical juxtaposition and the monoidal product by horizontal juxtaposition:

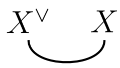
$\beta \circ \alpha =$ 

$\alpha \otimes \beta =$ 

.

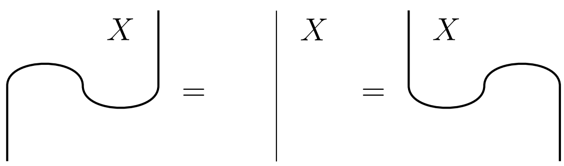
As the monoidal product is depicted by horizontal juxtaposition the associativity maps are implicitly used but not depicted. Similarly any strand labelled by the tensor identity is not drawn, therefore the unit isomorphisms are also implicitly used but not depicted. The maps cr_X and an_X are drawn as a cap and cup respectively:

$\mathrm{cr}_X =$ 

$\mathrm{an}_X =$ 

.

Note that this is consistent with horizontal juxtaposition depicting the monoidal product by Lemma 2.2.7. With graphical notation in hand we can rewrite conditions (2.1) and (2.2) as



. (2.3)

As all three diagrams are isotopic the graphical intuition behind dual objects is made clear.

DEFINITION 2.2.11. A *multifusion category* is a finite Schurian, monoidal, rigid category (for which the tensor product is linear) such that the tensor identity decomposes as follows,

$$\mathbf{1} = \bigoplus_{i \in \mathcal{I}} \mathbf{1}_i$$

where $\mathcal{I}$ is some indexing set and $\mathrm{Hom}(\mathbf{1}_i, \mathbf{1}_j) = \delta_{i,j}\,\mathbb{K}$. Furthermore, if $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{K}$ (i.e. $\#\mathcal{I} = 1$), $\mathcal{C}$ is called a *fusion category*.

REMARK 2.2.12. If $\mathbb{K}$ is algebraically closed then any finite, monoidal, rigid category is a multifusion category [EGNO15, Theorem 4.3.1.].

Multifusion categories get their name from the fact that they have certain fusion categories as summands. Indeed, we have the following,

PROPOSITION 2.2.13. *Let $\mathcal{C}$ be a multifusion category. Then $\mathcal{C}$ decomposes as*

$$\mathcal{C} = \bigoplus_{i,j \in \mathcal{I}} {}_i\mathcal{C}_j$$

where ${}_i\mathcal{C}_j = \mathbf{1}_i \otimes \mathcal{C} \otimes \mathbf{1}_j$.

Proof. These results all follow from Theorem 4.3.1. in [EGNO15].

□

REMARK 2.2.14. Theorem 4.3.1. in [EGNO15] shows that the proposition still holds if we drop the assumption that $\mathcal{C}$ has finitely many isomorphism classes of simple objects.

REMARK 2.2.15. The tensor product maps ${}_i\mathcal{C}_j \times {}_j\mathcal{C}_l$ to ${}_i\mathcal{C}_l$. Therefore, for all $i \in \mathcal{I}$, the category ${}_i\mathcal{C}_i$ is a fusion category.

2.2.2 | PIVOTAL STRUCTURE

DEFINITION 2.2.16. Let $\mathcal{C}$ be a rigid category. A *pivotal structure* on $\mathcal{C}$ is a choice of natural isomorphism

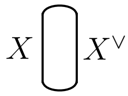
$$\delta_X : {}^{\vee}X \to X^{\vee}$$

such that (under the identification of Lemma 2.2.7)

$$\delta_{X\otimes Y} = \delta_Y \otimes \delta_X. \tag{2.4}$$

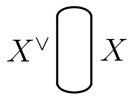
A rigid category equipped with a pivotal structure is called a *pivotal category*.

The map δ_X is suppressed from graphical notation; condition (2.4) guarantees that this doesn't cause inconsistencies. As a pivotal structure identifies left and right duals, we use $X^\vee$ to denote both. For example



(2.5)

is valid graphical notation for an element in $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ if $\mathcal{C}$ is pivotal but not if $\mathcal{C}$ is merely rigid. However, even in a pivotal category, (2.5) is not necessarily equal to



. (2.6)

DEFINITION 2.2.17. A pivotal category is called *spherical* if (2.5) and (2.6) define the same element of $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ for all X in $\mathcal{C}$. In this case said element is called the *dimension* of X and is denoted $d(X)$.

PROPOSITION 2.2.18. *Suppose $\mathcal{C}$ is a semisimple spherical category, $\mathrm{Irr}(\mathcal{C})$ is a complete set of simples in $\mathcal{C}$ and $S \in \mathrm{Irr}(\mathcal{C})$ is Schurian. Then $d(S)$ is an automorphism of $\mathbf{1}$.*

Proof. By Proposition 2.1.8 we have

$$S \otimes S^\vee = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, S\otimes S^\vee)\cdot \mathbf{1} \quad \oplus \bigoplus_{\substack{T\in \mathrm{Irr}(\mathcal{C})\\ \mathrm{Hom}_{\mathcal{C}}(\mathbf{1},T)=0}} \mathrm{Hom}_{\mathcal{C}}(T, S\otimes S^\vee)\cdot T$$

Let π be projection onto the $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, S\otimes S^\vee)\cdot \mathbf{1}$ summand and let i be the corresponding inclusion. As, by (2.2), cr_S and an_S are non zero and

$$\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, S\otimes S^\vee) = \mathrm{Hom}_{\mathcal{C}}(S\otimes S^\vee, \mathbf{1}) = \mathrm{End}_{\mathcal{C}}(S) = \mathbb{K}$$

$\pi\circ \mathrm{cr}_S$ and $\mathrm{an}_S \circ i$ are isomorphisms. As $d(S) = \mathrm{an}_S\circ \mathrm{cr}_S = \mathrm{an}_S \circ i\circ \pi\circ \mathrm{cr}_S$ this concludes the proof.

□

DEFINITION 2.2.19. The *dimension* of a finite spherical category $\mathcal{C}$ is given by

$$d(\mathcal{C}) := \sum_{S \in \mathrm{Irr}(\mathcal{C})} d(S)^2.$$

2.2.3 | THE BRAIDING AND NON-DEGENERACY

DEFINITION 2.2.20. Let $\mathcal{C}$ be a monoidal category. A *braiding* on $\mathcal{C}$ is a collection of natural isomorphisms $\sigma_{X,Y} \colon X \otimes Y \to Y \otimes X$, such that the diagrams

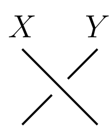
$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "\sigma_{X,Y\otimes Z}"] \arrow[dr, "\sigma_{X,Y}\otimes \mathbf{1}"'] & & Y \otimes Z \otimes X \\ & Y \otimes X \otimes Z \arrow[ur, "\mathbf{1}\otimes\sigma_{X,Z}"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[rr, "\sigma_{X\otimes Y,Z}"] \arrow[dr, "\mathbf{1}\otimes\sigma_{Y,Z}"'] & & Y \otimes Z \otimes X \\ & Y \otimes X \otimes Z \arrow[ur, "\sigma_{X,Z}\otimes \mathbf{1}"'] & \end{tikzcd}$$

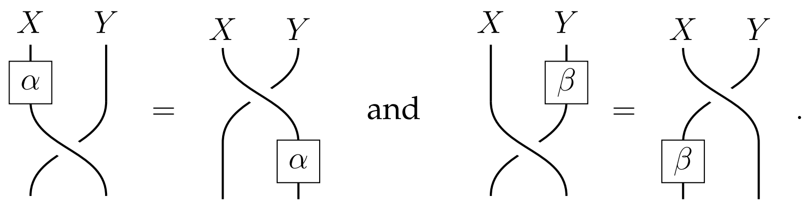
commute. These conditions are often referred to as the *hexagon identities* (our diagrams are triangular as we have suppressed the associativity isomorphisms).

A monoidal category equipped with a braiding is said to be *braided*. In graphical notation this braiding is depicted by the over-crossing,

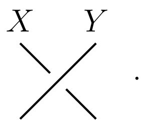


.

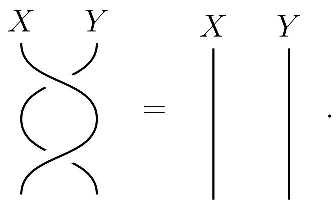
The hexagon identities guarantee that this notation is consistent with horizontal juxtaposition depicting the monoidal product. Naturality of the braiding allows morphisms to pass over and under strands, i.e.



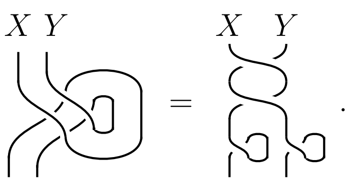
If σ is a braiding on a monoidal category $\mathcal{C}$ then $\overline{\sigma}$, given by $\overline{\sigma}_{X,Y} := (\sigma^{-1}_{Y,X})$, also defines a braiding on $\mathcal{C}$ called the *opposite* braiding to σ. In graphical notation the opposite braiding is depicted by the under-crossing,



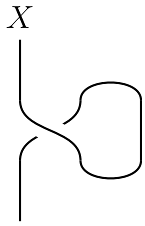
We therefore have the "Reidemeister II" rule



This, together with (2.3) (which resembles a "Reidemeister 0" rule), makes one wonder whether, for a spherical braided category, graphical notation is well defined up to tangle isotopy. For this to be possible we have to assume a certain compatibility between the braiding and the pivotal structure which may be described graphically as



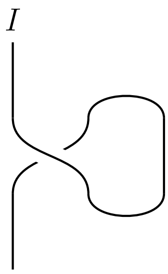
A braiding which satisfies this condition is called balanced; for the remainder of this thesis we suppose that all braidings on spherical categories are balanced. Even when the braiding is balanced, however, graphical notation is not well defined up to tangle isotopy as, in general,

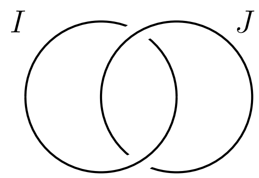


does not define $\mathrm{id}_X \in \mathrm{End}_{\mathcal{C}}(X)$ and so there is no "Reidemeister I" rule. Dropping the "Reidemeister I" rule gives us the notion of *ribbon tangle isotopy* and it is true that graphical notation is well defined up to ribbon tangle isotopy [RT90]. This motivates the following terminology.

DEFINITION 2.2.21. A spherical (balanced) braided category is called a *ribbon* category.

We have now introduced all the structure required to define a modular tensor category; however, one non-degeneracy property remains to be explained. Let $\mathcal{C}$ be a semisimple ribbon category and let $\mathrm{Irr}(\mathcal{C})$ be a complete set of simples. We suppose that **1** is simple ($\mathcal{C}$ is therefore a fusion category) and that all simples in $\mathcal{C}$ are Schurian. We consider the following $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$ matrices,

$\mathcal{T}_{IJ} := \delta_{I,J}$ 

$\mathcal{S}_{IJ} :=$ 

. (2.7)

REMARK 2.2.22. If we replace the I strand in the definition of $\mathcal{T}_{IJ}$ with an $I^\vee$ strand we obtain an equivalent definition [EGNO15, Section 8.10.]. Similarly if we replace the I and J strands in the definition of $\mathcal{S}_{IJ}$ with $I^\vee$ and $J^\vee$ strands respectively we also obtain an equivalent definition [EGNO15, Remark 8.1.12.3].

These matrices are called the T-matrix and the S-matrix respectively. Collectively they are know are the *modular data* of $\mathcal{C}$.

DEFINITION 2.2.23. A *modular tensor category* (MTC) is a finite ribbon category such that

(1) The tensor identity is simple and all simples are Schurian.

(2) The category has non-zero dimension.

(3) The above defined $\mathcal{S}$ and $\mathcal{T}$ matrices are invertible, i.e. the modular data is non-singular.

A finite ribbon category that satisfies condition (1) and (2) but not necessarily condition (3) is called a *pre-modular tensor category* (PTC).

REMARK 2.2.24. By [EGNO15, Proposition 8.20.16.] condition (2) is automatically satisfied if $\mathbb{K}$ is algebraically closed.

REMARK 2.2.25. Modular tensor categories are so-called as their modular data satisfies the following equations,

$$(\mathcal{S}\mathcal{T})^3 = \lambda \mathcal{S}^2,\ \mathcal{S}^2 = d(\mathcal{C})\, C,\ \mathcal{T}C = C\mathcal{T}$$

where $\lambda \in \mathbb{K}$ and $C := (\delta_{I^\vee,J})_{I,J\in\mathrm{Irr}(\mathcal{C})}$ is the *charge conjugation matrix* [EGNO15, Section 8.16.]. As $C^2 = \mathrm{id}$ these equations imply that the modular data gives a projective representation of $\mathrm{SL}_2(\mathbb{Z})$ a.k.a. the *modular* group.

REMARK 2.2.26. From Remark 2.2.22 we see that $\mathcal{S}$ and $\mathcal{T}$ commute with the charge conjugation matrix even when $\mathcal{C}$ is only assumed to be pre-modular.

2.3 | GRAPHICAL PROPOSITIONS

2.3.1 | PERFECT PAIRINGS

Let $\mathcal{C}$ be a Schurian (see Section 2.1.1) semisimple category and let $\mathrm{Irr}(\mathcal{C})$ be a complete set of simples in $\mathcal{C}$.

PROPOSITION 2.3.1. *Let R be in $\mathrm{Irr}(\mathcal{C})$ and let X be in $\mathcal{C}$. The pairing*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(R,X)\otimes\mathrm{Hom}_{\mathcal{C}}(X,R) &\to \mathbb{K}\\ f\otimes g &\mapsto g\circ f\end{aligned}$$

is perfect.

Proof. By semisimplicity we have

$$X = \bigoplus_{i \in \mathcal{I}} X_i$$

where the X_i are simple objects and $\mathcal{I}$ is an indexing set. We consider the subset $J := \{i \in I \mid X_i \cong R\} \subset I$. Then we have

$$\mathrm{Hom}_{\mathcal{C}}(R, X) \cong \mathbb{K}^J \quad \text{and} \quad \mathrm{Hom}_{\mathcal{C}}(X, R) \cong \mathbb{K}^J\,.$$

As composition is given by the standard pairing, which is perfect, we are done.

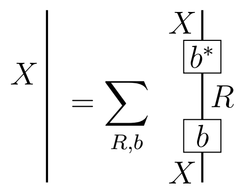
□

DEFINITION 2.3.2. Let R be in $\mathrm{Irr}(\mathcal{C})$ and let X be in $\mathcal{C}$. For every basis $\{b\} \subset \mathrm{Hom}_{\mathcal{C}}(R, X)$ we use $\{b^*\}$ to denote the dual basis of $\mathrm{Hom}_{\mathcal{C}}(X, R)$ with respect to the perfect pairing given by Proposition 2.3.1.

The following conventions will be used throughout this thesis. They will be of particular importance for the graphical computations.

- Unless otherwise specified, a sum over a variable object in $\mathcal{C}$ ranges over $\mathrm{Irr}(\mathcal{C})$.
- Unless otherwise specified, a sum over a variable morphism in $\mathcal{C}$ ranges over a basis of the appropriate Hom-space.

LEMMA 2.3.3. *Let X be in $\mathcal{C}$. We have*

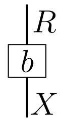


(2.8)

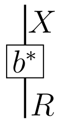
where, as per the above conventions, R ranges over a basis of $\mathrm{Irr}(\mathcal{C})$ *and b ranges over a basis of* $\mathrm{Hom}_{\mathcal{C}}(R, X)$.

Proof. By Proposition 2.1.8, we have a natural identification $\bigoplus_R \mathrm{Hom}_{\mathcal{C}}(R, X) \otimes R = X$. Using the bases $\{b\}$, for each $\mathrm{Hom}_{\mathcal{C}}(R, X)$, we get an isomorphism

$f\colon \bigoplus_{R,b} R \to X$ given diagrammatically by

$f_{R,b} =$  .

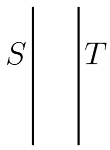
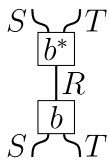
On the other hand, the map $g\colon X \to \bigoplus_{R,b} R$ given by

$g_{R,b} =$ 

satisfies $g \circ f = \mathrm{id}$ and so g is the inverse to f. The right-hand side of (2.8) is simply $f \circ g$ and hence equal to the identity, as required.

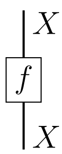
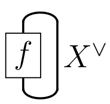
$\square$

REMARK 2.3.4. When $\mathcal{C}$ is monoidal, we will often use the following instance of Lemma 2.3.3. For $S, T \in \mathrm{Irr}(\mathcal{C})$ we have

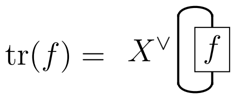
 $= \sum_{R,b}$  .

We now additionally suppose that $\mathcal{C}$ is monoidal and rigid. We also equip it with a spherical pivotal structure (see Section 2.2) and suppose that $\mathbf{1}$ is a simple object in $\mathcal{C}$. In other words, $\mathcal{C}$ is a spherical fusion category, see Definition 2.2.11 and 2.2.17.

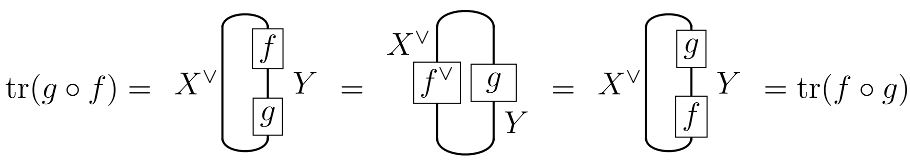
DEFINITION 2.3.5. Let f be in $\mathrm{End}_{\mathcal{C}}(X)$ for some object X in $\mathcal{C}$. The *trace* of f is defined by

$\mathrm{tr}:$  $\longmapsto$  $\in \mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \mathbb{K}\,.$

We note that, as $\mathcal{C}$ is spherical, the trace is also given by



and that taking the trace of a composition is commutative i.e.



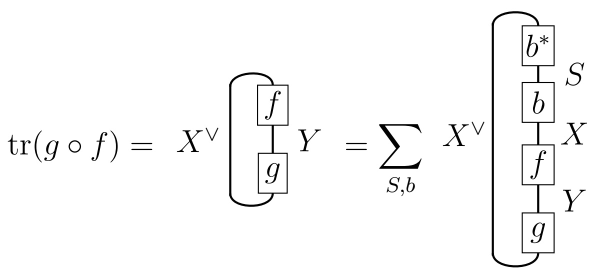
for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

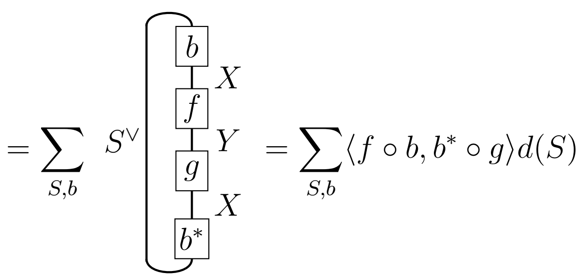
PROPOSITION 2.3.6. *The pairing*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, X) &\to \mathbb{K} \\ f \otimes g &\mapsto \mathrm{tr}(g \circ f) \end{aligned}$$

is perfect.

Proof. Applying Proposition 2.3.3 to X gives us





where $\langle -, - \rangle$ denotes the perfect pairing defined in Proposition 2.3.1. Let $f \in$

$\mathrm{Hom}_{\mathcal{C}}(X, Y)$ be non zero. Then there exists $S \in \mathrm{Irr}(\mathcal{C})$ and $b \in \mathrm{Hom}_{\mathcal{C}}(S, X)$ such that $f \circ b \neq 0$. As $\langle -, - \rangle$ is perfect there exists $g_S \in \mathrm{Hom}_{\mathcal{C}}(Y, S)$ such that $\langle f \circ b, g_S \rangle \neq 0$. Then we have

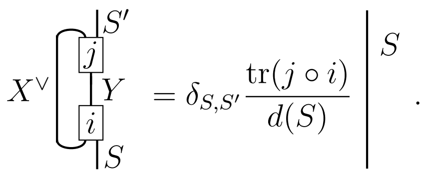
$$\mathrm{tr}(f, b \circ g_S) = \langle f \circ b, b \circ b^* \circ g_S \rangle d(S) = \langle f \circ b, g_S \rangle d(S) \neq 0.$$

□

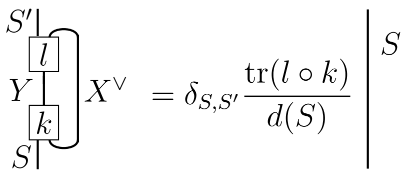
2.3.2 | TWISTED TENSOR DECOMPOSITION

As in the previous section, let $\mathcal{C}$ be a spherical fusion category. Much of the work carried out in this thesis will be done relative to a fixed spherical fusion category $\mathcal{C}$, and from now on, when taking tensor products in this category, we will omit the "$\otimes$" symbol and write XY for $X \otimes Y$. However, we will continue to write the "$\otimes$" symbol when taking a tensor product in any other category.

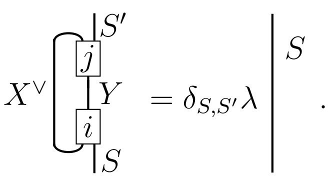
LEMMA 2.3.7. *Let X and Y be in $\mathcal{C}$. For all $S, S' \in \mathrm{Irr}(\mathcal{C})$, $i \in \mathrm{Hom}_{\mathcal{C}}(Y, XS)$ and $j \in \mathrm{Hom}_{\mathcal{C}}(XS', Y)$, we have*



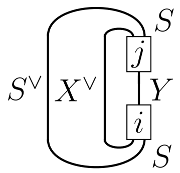
We also have, for any $k \in \mathrm{Hom}_{\mathcal{C}}(Y, SX)$ and $l \in \mathrm{Hom}_{\mathcal{C}}(S'X, Y)$,

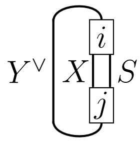


Proof. As S and S' are simple objects we have



for some $\lambda \in \mathbb{K}$. To compute λ we suppose $\delta_{S,S'} = 1$ and take the trace to obtain

$\lambda d(S) =$ 

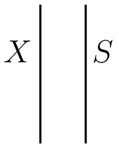
$=$ 

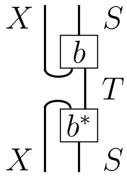
$= \mathrm{tr}(j \circ i)$

Therefore $\lambda = \frac{\mathrm{tr}(j \circ i)}{d(S)}$. This proves the first part of the lemma, the second part is proved analogously.

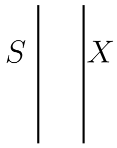
□

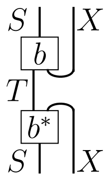
LEMMA 2.3.8. *Let X be in $\mathcal{C}$ and S be in $\mathrm{Irr}(\mathcal{C})$. We have*

$d(S)$ 

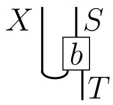
$= \sum_{T,b} d(T)$ 

and

$d(S)$ 

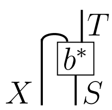
$= \sum_{T,b} d(T)$ 

.

Proof. The map $\alpha(b) =$ 

is the image of b under the adjunction

$$\alpha\colon \mathrm{Hom}_{\mathcal{C}}(S, X^\vee T) \to \mathrm{Hom}_{\mathcal{C}}(XS, T)$$

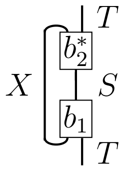
and similarly the map $\beta(b^*)\colon =$ 

is the image of b^* under the adjunction

$$\beta\colon \mathrm{Hom}_{\mathcal{C}}(X^\vee T, S) \to \mathrm{Hom}_{\mathcal{C}}(T, XS).$$

Therefore as b ranges over a basis of $\mathrm{Hom}_{\mathcal{C}}(S, X^\vee T)$, $\alpha(b)$ ranges over a basis $\mathcal{G}$ of $\mathrm{Hom}_{\mathcal{C}}(XS, T)$ and $\beta(b^*)$ ranges over a basis $\mathcal{H}$ of $\mathrm{Hom}_{\mathcal{C}}(T, XS)$. However $\mathcal{G}$ and

$\mathcal{H}$ are *not* dual to one another. Indeed evaluating $\alpha(b_1) \in \mathcal{G}$ on $\beta(b_2^*) \in \mathcal{H}$ and applying Lemma 2.3.7 gives



$$= \frac{\mathrm{tr}(b_2^* \circ b_1)}{d(T)} \Bigg|\, T = \delta_{b_1,b_2} \frac{\mathrm{tr}(id_S)}{d(T)} \Bigg|\, T = \delta_{b_1,b_2} \frac{d(S)}{d(T)} \Bigg|\, T .$$

This implies that $\mathcal{G}$ and $\mathcal{H}$ are *pseudo-dual*, to make them truly dual we would have to rescale one of them by $\frac{d(T)}{d(S)}$. This, together with Lemma 2.3.3, proves the first part of the lemma, the second part is proved analogously.

□

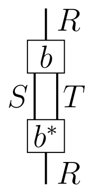
REMARK 2.3.9. An alternative proof of the above lemma may be found in [Kon08, Lemma 5.1].

LEMMA 2.3.10. *Let* $\hom_{\mathcal{C}}(X, Y)$ *denote the dimension of* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *and let* R *be in* $\mathrm{Irr}(\mathcal{C})$. *Then*

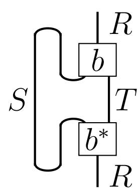
$$\sum_{S,T} \hom_{\mathcal{C}}(R, ST) d(S) d(T) = d(R) d(\mathcal{C}).$$

Proof. For $\lambda = \sum_{S,T} \hom_{\mathcal{C}}(R, ST) d(S) d(T)$ we have

$$\lambda \Bigg|\, R = \sum_{S,T,b} d(S) d(T)$$



$$= \sum_{S,T,b} d(S) d(T)$$



$$= d(R) d(\mathcal{C}) \Bigg|\, R$$

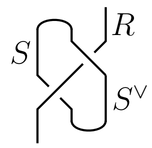
where the final equality uses Lemma 2.3.8. □

2.3.3 | The Killing Ring

We now suppose that $\mathcal{C}$ is an MTC.

PROPOSITION 2.3.11 (Killing Ring). *Let R be in $\mathrm{Irr}(\mathcal{C})$. Then*

$$\sum_S d(S)$$



$$= \delta_{R,\mathbf{1}} d(\mathcal{C})$$

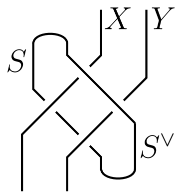
where **1** *is the tensor identity.*

Proof. See Corollary 3.1.11. in [BK01].

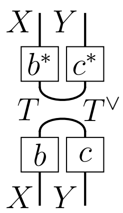
□

COROLLARY 2.3.12. *Let X and Y be in $\mathcal{C}$. Then*

$$\sum_S d(S)$$



$$= d(\mathcal{C}) \sum_{T,b,c} \frac{1}{d(T)}$$



.

Proof. This follows immediately from Lemma 2.3.3, Proposition 2.3.11 and the fact that, for $R, T \in \mathrm{Irr}(\mathcal{C})$, we have

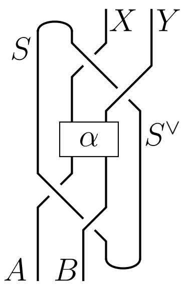
$$\mathrm{Hom}_{\mathcal{C}}(RT, \mathbf{1}) = \mathrm{Hom}_{\mathcal{C}}(T, R^\vee) = \delta_{R^\vee,T}\, \mathrm{id}_{R^\vee} .$$

We note that the $d(R)^{-1}$ term appears as the creation and annihilation morphisms are *not* dual to one another, indeed they compose to the dimension. To make them dual we therefore weight by the inverse of the dimension.

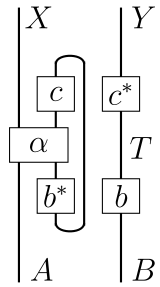
□

PROPOSITION 2.3.13. *We consider X, Y, A, B in $\mathcal{C}$, and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(IJ, XY)$. Then*

$$\sum_S d(S)$$

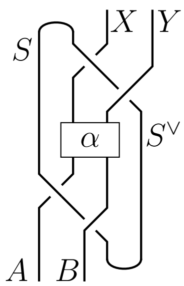


$$= d(\mathcal{C}) \sum_{T,b,c} \frac{1}{d(T)}$$

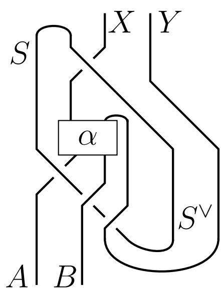


Proof. We have

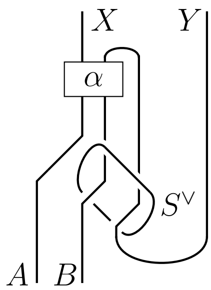
$$\sum_S d(S)$$



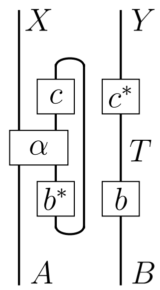
$$= \sum_S d(S)$$



$$= \sum_S d(S)$$



$$= d(\mathcal{C}) \sum_{T,b,c} \frac{1}{d(T)}$$



where the final equality uses Corollary 2.3.12.

$\square$

CHAPTER 3

THE TUBE CATEGORY

The main goal of this chapter is to introduce and study the tube category of a spherical fusion category. This constitutes one of the fundamental constructions in this thesis.

Section 3.1 first defines the tube category of a spherical fusion category, which may be thought of as a categorical analogue of Oceanu's tube algebra [Ocn94]. This definition has previously appeared in [HK19]. In the case when $\mathcal{C}$ is modular, we introduce, for X and Y in $\mathcal{C}$, an idempotent ϵ_X^Y. The corresponding Hom-spaces are subsequently computed. In particular it is shown that, when $\mathcal{C}$ is modular, the set $\{\epsilon_I^J\}_{I,J\in\mathrm{Irr}(\mathcal{C})}$ forms a complete set of orthogonal primitive idempotents in $\mathcal{TC}$. Finally, the fact that the endomorphism algebra of the tensor identity in the tube category is canonically identified with the complexified Grothendieck ring is exploited to prove that said ring is semisimple and compute its spectrum.

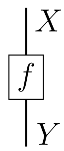
Section 3.2 analyses the the category of representations of $\mathcal{TC}$ (denoted $\mathcal{RTC}$). When $\mathcal{C}$ is a modular tensor category, $\mathcal{RTC}$ may also be thought of as the idempotent completion of $\mathcal{TC}$. In particular this section characterises the data required to extend a representation of $\mathcal{C}$ to a representation of $\mathcal{TC}$.

Section 3.3 starts by describing the centre of $\mathcal{C}$, denoted $Z(\mathcal{C})$, a category whose objects consist of objects in $\mathcal{C}$ together with a half-braiding. An equivalence between $\mathcal{RTC}$ and $Z(\mathcal{C})$ is then described. This equivalence stems from the fact that, when $\mathcal{C}$ is a spherical fusion category, the Yoneda embedding is an equivalence and that the data required to extend the image of X under the Yoneda embedding to $\mathcal{TC}$ corresponds to a half braiding on X. A tensor functor Φ from $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ to $Z(\mathcal{C}) = \mathcal{RTC}$ is then described. It is shown that Φ maps $X \boxtimes Y$ in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ to the functor represented by the idempotent ϵ_X^Y. This, together with the results of Section 3.1, recovers a theorem due to Müger [Müg03] that Φ is an equivalence when $\mathcal{C}$ is modular.

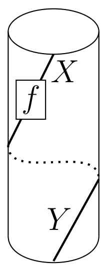
3.1 | Introducing the Tube Category

3.1.1 | Definition of the Tube Category

Let $\mathcal{C}$ be a spherical fusion category, see Definition 2.2.11 and 2.2.17. The *tube category*, denoted $\mathcal{TC}$, shares the same objects as $\mathcal{C}$ but has more morphisms i.e. $\mathrm{Hom}_{\mathcal{C}}(X, Y) \leq \mathrm{Hom}_{\mathcal{TC}}(X, Y)$. The intuition is that whereas morphisms in $\mathcal{C}$ may be represented graphically as diagrams drawn on a bounded region of the plane, morphisms in $\mathcal{TC}$ are given by diagrams drawn on a *cylinder*. For example, for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ diagrammatically represented by

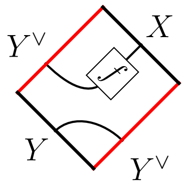


there will be a morphism in $\mathcal{TC}$ diagrammatically represented by



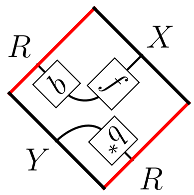
. (3.1)

We capture such morphisms by drawing diagrams in a diamond and glueing the upper left and lower right edges. For example morphism (3.1) is represented by



.

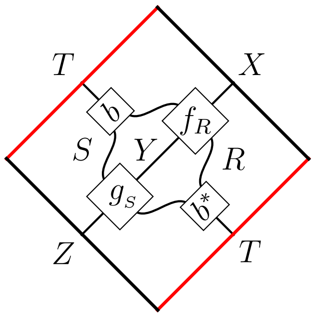
We note that this diagram may also be read vertically and interpreted as an element in $\mathrm{Hom}_{\mathcal{C}}(Y^\vee X, YY^\vee)$. We also note that due to Lemma 2.3.3 we may restrict ourselves to only gluing *simple* strands. In this way morphism (3.1) would be represented as

$\sum_{R,b}$ 

where b ranges over a basis of $\mathrm{Hom}_{\mathcal{C}}(R, Y^\vee)$. We note that each diagram may now be read vertically as an element in $\mathrm{Hom}_{\mathcal{C}}(RX, YR)$. With this motivation in mind we may proceed with the definition of $\mathcal{TC}$.

DEFINITION 3.1.1. Let $\mathcal{C}$ be a spherical fusion category. The associated *tube category*, denoted $\mathcal{TC}$, is defined as the following category,

1. $\mathrm{Obj}(\mathcal{TC}) := \mathrm{Obj}(\mathcal{C})$
2. $\mathrm{Hom}_{\mathcal{TC}}(X, Y) := \bigoplus_R \mathrm{Hom}_{\mathcal{C}}(RX, YR)$
3. Let f be in $\mathrm{Hom}_{\mathcal{TC}}(X, Y)$ and let g be in $\mathrm{Hom}_{\mathcal{TC}}(Y, Z)$. We define $g \circ f$ as follows (using the diagrams explained above)

$g \circ f := \bigoplus_T \sum_{S,R,b}$  (3.2) where f_R and g_S are the $\mathrm{Hom}_{\mathcal{C}}(RX, YR)$ and $\mathrm{Hom}_{\mathcal{C}}(SY, ZS)$ components of f and g respectively and b ranges over a basis of $\mathrm{Hom}_{\mathcal{C}}(T, SR)$. We note that $g \circ f \in \bigoplus_T \mathrm{Hom}_{\mathcal{C}}(TX, ZT) = \mathrm{Hom}_{\mathcal{TC}}(X, Z)$ as desired.

From Lemma 2.3.3 we see that this definition agrees with the intuition that composition corresponds to vertically stacking the cylinders upon which the diagrams are drawn. This intuition, together with the associativity of the tensor product, makes it clear that composition in $\mathcal{TC}$ is associative.

REMARK 3.1.2. At this point, a careful reader reader might protest that the tensor product is merely weakly associative and yet composition in a category must be strongly associative. However, this is not an issue as the associator isomorphisms will simply modify the basis appearing in (3.2) leaving the composition unchanged.

REMARK 3.1.3. The summand indexed by **1** in $\mathrm{Hom}_{\mathcal{TC}}(X, Y)$ is $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. This gives a map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{TC}}(X, Y)$ such that

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,Z) \arrow[r, hook] \arrow[d, "\circ"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{TC}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{TC}}(Y,Z) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[r, hook] & \mathrm{Hom}_{\mathcal{TC}}(X,Z)
\end{tikzcd}$$

commutes. In other words $\mathcal{C}$ is a *subcategory* of $\mathcal{TC}$. In particular the identity in $\mathrm{End}_{\mathcal{TC}}(X)$ is given by the image of $\mathrm{id}_X \in \mathrm{End}_{\mathcal{C}}(X)$ under this embedding.

REMARK 3.1.4. If we consider the algebra

$$\mathcal{TA} := \mathrm{End}_{\mathcal{TC}}\left(\bigoplus_S S\right)$$

we recover Oceanu's *tube algebra* [Ocn94]. As $\bigoplus\limits_S S$ is a projective generator in $\mathcal{TC}$

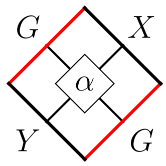
the functor

$$\begin{aligned} \mathcal{RTC} &\to \text{Mod-}\mathcal{TA} \\ F &\mapsto \mathrm{Hom}_{\mathcal{RTC}}\left(F, \bigoplus_S S\right) \end{aligned}$$

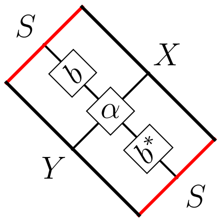
gives an equivalence, i.e. $\mathcal{TC}$ is Morita equivalent to $\mathcal{TA}$.

REMARK 3.1.5. The definition of Hom-spaces in $\mathcal{TC}$ has the following interesting consequence. Let $\mathcal{K}(\mathcal{C})$ denote the Grothendieck ring of $\mathcal{C}$ and let $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$ denote $\mathcal{K}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{K}$ (we call $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$ the *complexified Grothendieck ring* even though $\mathbb{K}$ is not necessarily the complex numbers). Then $\mathrm{End}_{\mathcal{TC}}(\mathbf{1})$ and $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$ are canonically isomorphic algebras. Indeed, $\mathrm{End}_{\mathcal{TC}}(\mathbf{1}) = \bigoplus_S \mathrm{End}(S) = \bigoplus_S \mathbb{K}$ is precisely the underlying vector space of $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$. Furthermore, composition in $\mathrm{End}_{\mathcal{TC}}(\mathbf{1})$ corresponds to the tensor product in $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$ by Lemma 2.3.3.

For X, Y, G in $\mathcal{C}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(GX, YG)$ we use

$\alpha_G =$ 

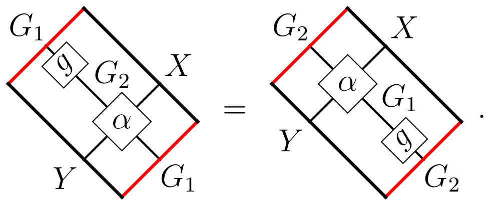
as shorthand for

$\bigoplus_S \sum_b$ 

$\in \bigoplus_S \mathrm{Hom}_{\mathcal{C}}(SX, YS) = \mathrm{Hom}_{\mathcal{TC}}(X, Y).$

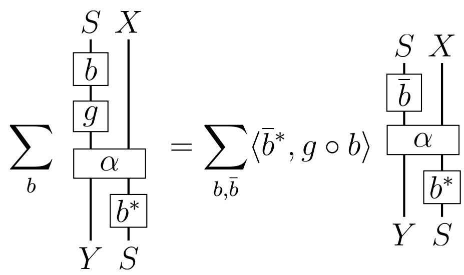
REMARK 3.1.6. This new notation may potentially create confusion with the pre-existing convention that, for $f \in \mathrm{Hom}_{\mathcal{TC}}(X, Y)$ and $R \in \mathrm{Irr}(\mathcal{C})$, f_R denotes the $\mathrm{Hom}_C(RX, YR)$ competent of f. To avoid such confusion we will restrict our new notation to the letters G and H.

We note that, using this new notation, we have

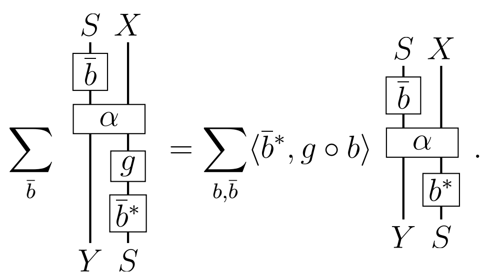


(3.3)

Indeed the S-summand of the left hand side of (3.3) is



and similarly the right hand side of (3.3) is



This serves as an effective reality-check that Definition 3.1.1 captures our original motivation of constructing an annular analogue of $\mathcal{C}$.

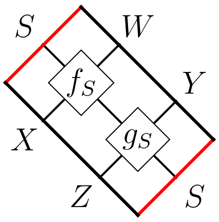
However, describing $\mathcal{TC}$ as an annular analogue of $\mathcal{C}$ may be a little misleading, depending on one's understanding of a cylinder. The important subtlety is that diagrams representing morphisms in $\mathcal{TC}$ are drawn on $\mathcal{S}^1 \times \mathcal{I}$ where $\mathcal{I}$ is an interval and S^1 is a circle *together with a distinguished point*. This allows us to glue the cylinders together, giving the following associative product.

DEFINITION 3.1.7. Let $\otimes_{\mathcal{TC}} \colon \mathcal{TC} \times \mathcal{TC} \to \mathcal{TC}$ be the bifunctor given by

$$X \otimes_{\mathcal{TC}} Y = XY$$

for X, Y in $\mathcal{TC}$ and

$$f \otimes_{\mathcal{TC}} g = d(\mathcal{C}) \bigoplus_S \frac{1}{d(S)}$$



$$\in \mathrm{Hom}_{\mathcal{TC}}(WY, XZ) \tag{3.4}$$

for $f \in \mathrm{Hom}_{\mathcal{TC}}(W, X)$ and $g \in \mathrm{Hom}_{\mathcal{TC}}(Y, Z)$.

We note that this product does not give a monoidal product as there is no unit. The tensor identity $\mathbf{1}$ in $\mathcal{C}$ fails to give a unit as the functor

$$- \otimes_{\mathcal{TC}} \mathbf{1} \colon \mathcal{TC} \to \mathcal{TC}$$

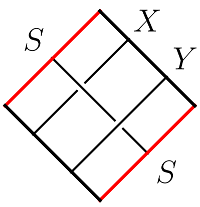
maps $\alpha \in \mathrm{Hom}_{\mathcal{TC}}(X, Y)$ to $d(\mathcal{C})\alpha_{\mathbf{1}} \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and so the unit isomorphisms fail to be natural.

REMARK 3.1.8. The $d(\mathcal{C})$ and $\frac{1}{d(S)}$ factors appearing in (3.4) may seem a little mysterious. They will eventually be justified by Propsition 3.3.5.

3.1.2 | IDEMPOTENTS IN $\mathcal{TC}$

Let $\mathcal{C}$ be a pre-modular tensor category (PTC) and let X, Y be objects in $\mathcal{C}$. The following morphisms will be of great importance throughout this thesis.

$$\epsilon_X^Y = \frac{1}{d(\mathcal{C})} \bigoplus_S d(S)$$



$$\in \mathrm{End}_{\mathcal{TC}}(XY).$$

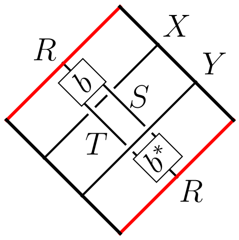
PROPOSITION 3.1.9. *We have*

$$\epsilon_X^Y \circ \epsilon_X^Y = \epsilon_X^Y$$

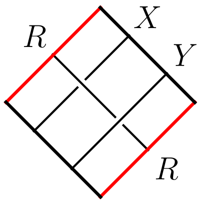
i.e. ϵ_X^Y *is an idempotent.*

Proof. We compute

$$\epsilon_X^Y \circ \epsilon_X^Y = \frac{1}{d(\mathcal{C})^2} \bigoplus_R \sum_{S,T,b} d(S)d(T)$$



$$= \frac{1}{d(\mathcal{C})^2} \bigoplus_R \left(\sum_{S,T} \hom(R, ST) d(S) d(T) \right)$$



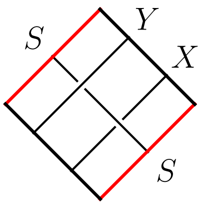
$$= \epsilon_X^Y$$

where the final equality uses Lemma 2.3.10.

∎

REMARK 3.1.10. The notation ϵ_X^Y is chosen (as opposed to ϵ_{XY}) as $(XY, \epsilon_X^Y)^\sharp$ is isomorphic (as an object in $\mathcal{RTC}$) to $(YX, \tilde{\epsilon}_X^Y)^\sharp$, where

$$\tilde{\epsilon}_X^Y = \frac{1}{d(\mathcal{C})} \bigoplus_S d(S)$$



$$\in \mathrm{End}_{\mathcal{TC}}(YX). \tag{3.5}$$

The isomorphism is in fact given by the embedding of the braiding on $\mathcal{C}$ into $\mathcal{TC}$. Therefore the isomorphism class of ϵ_X^Y is really determined by the fact that the X strand is *under*-braided and the Y strand is *over*-braided. This motivates the notation.

REMARK 3.1.11. The notation ϵ_X^Y is also well behaved with respect to the associative product given by Definition 3.1.7, as we have

$$(XYAB, \epsilon_X^Y \otimes_{\mathcal{TC}} \epsilon_A^B)^\sharp = (XAYB, \epsilon_{XA}^{YB})^\sharp$$

where the isomorphism is once again given by the braiding.

PROPOSITION 3.1.12. *We have*

$$\mathrm{Hom}_{\mathcal{C}}(Z, XY) = \mathrm{Hom}_{\mathcal{TC}}(Z, e_X^Y) \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{C}}(XY, Z) = \mathrm{Hom}_{\mathcal{TC}}(e_X^Y, Z)$$

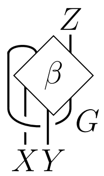
where, as in Section 2.1.2, $\mathrm{Hom}_{\mathcal{TC}}(Z, e_X^Y) = \{\beta \in \mathrm{Hom}_{\mathcal{TC}}(Z, XY) \mid \epsilon_X^Y \circ \beta = \beta\}$.

Proof. We consider the maps

$$\begin{aligned} \phi_Z \colon \mathrm{Hom}_{\mathcal{C}}(Z, XY) &\to \mathrm{Hom}_{\mathcal{TC}}(Z, e_X^Y) \\ \alpha &\mapsto \epsilon_X^Y \circ \alpha \end{aligned}$$

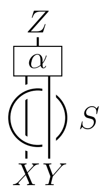
and

$$\varphi_Z \colon \mathrm{Hom}_{\mathcal{TC}}(Z, e_X^Y) \to \mathrm{Hom}_{\mathcal{C}}(Z, XY)$$

$$\beta_G \mapsto$$ 

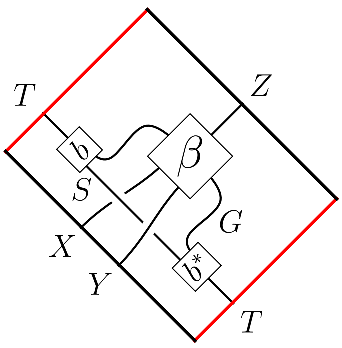
.

We have

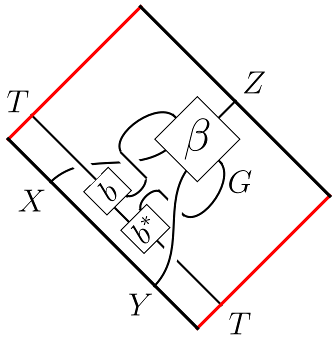
$$\varphi_Z \circ \phi_Z(\alpha) = \frac{1}{d(\mathcal{C})} \sum_S d(S)$$ 

$$= \alpha$$

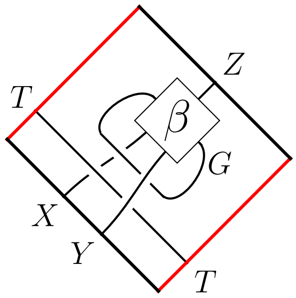
and, for $\beta_G \in \mathrm{Hom}_{\mathcal{TC}}(Z, \epsilon_X^Y)$,

$$\beta_G = \epsilon_X^Y \circ \beta_G = \frac{1}{d(\mathcal{C})} \bigoplus_T \sum_{S,b} d(S)$$ 

$$= \frac{1}{d(\mathcal{C})} \bigoplus_T \sum_{S,b} d(S)$$



$$= \frac{1}{d(\mathcal{C})} \bigoplus_T d(T)$$



$$= \phi_Z \circ \varphi_Z(\beta_G)$$

where the penultimate equality is due to Lemma 2.3.8. Therefore ϕ_Z and φ_Z are inverse to one another. This proves the first equality, the second may be proved analogously.

□

When $\mathcal{C}$ is a modular tensor category (MTC) we can describe the morphisms between these idempotents.

THEOREM 3.1.13. *Let $\mathcal{C}$ be an MTC. We consider X, Y, A, B in $\mathcal{C}$. We have*

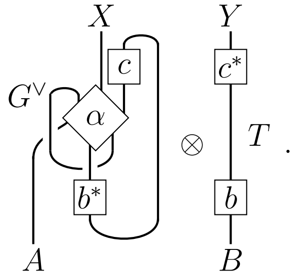
$$\mathrm{Hom}_{\mathcal{TC}}(\epsilon_X^Y, \epsilon_A^B) = \mathrm{Hom}_{\mathcal{C}}(X, A) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, B).$$

Proof. We consider the maps

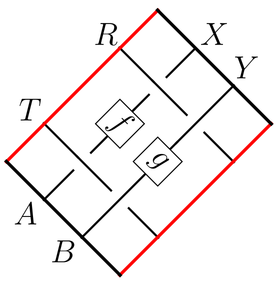
$$\begin{aligned} \phi\colon\ \mathrm{Hom}_{\mathcal{C}}(X, A) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, B) &\to \mathrm{Hom}_{\mathcal{TC}}(\epsilon_X^Y, \epsilon_A^B) \\ f \otimes g &\mapsto \epsilon_A^B \circ (f \otimes g) \circ \epsilon_X^Y \end{aligned}$$

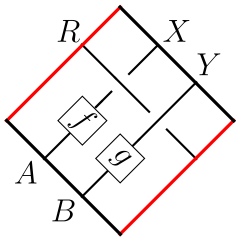
and

$$\varphi\colon \operatorname{Hom}_{\mathcal{TC}}(\epsilon_X^Y, \epsilon_A^B) \to \operatorname{Hom}_{\mathcal{C}}(X, A) \otimes \operatorname{Hom}_{\mathcal{C}}(Y, B)$$

$$\alpha_G \mapsto \sum_{T,b,c} \frac{1}{d(T)}$$


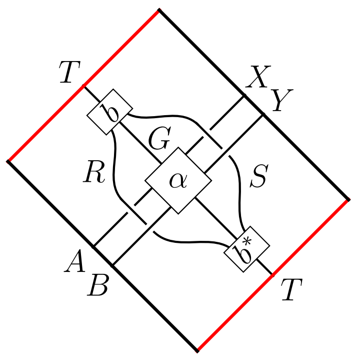
We have

$$\varphi \circ \phi(f \otimes g) = \varphi \left(\frac{1}{d(\mathcal{C})^2} \sum_{R,T} d(R)d(T) \right.$$


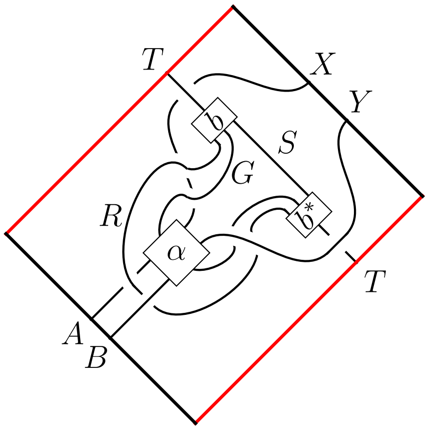
$$= \varphi \left(\frac{1}{d(\mathcal{C})} \sum_{R,T} d(R) \right.$$


$$= \sum_{T,b,c} b^*(g \circ c)\, f \otimes (b \circ c^*) = f \otimes g$$

and, for $\alpha_G \in \operatorname{Hom}_{\mathcal{TC}}(\epsilon_X^Y, \epsilon_A^B)$,

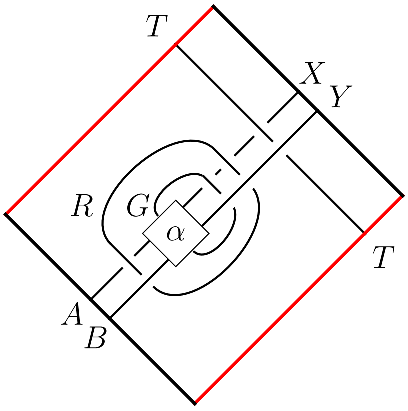
$$\alpha_G = \epsilon_A^B \circ \alpha_G \circ \epsilon_X^Y = \frac{1}{d(\mathcal{C})^2} \bigoplus_T \sum_{R,S,b} d(R)d(S)$$


$$= \frac{1}{d(\mathcal{C})^2} \bigoplus_T \sum_{R,S,b} d(R)d(S)$$



then, by Lemma 2.3.8, we have

$$= \frac{1}{d(\mathcal{C})^2} \bigoplus_T \sum_{R,b} d(R)d(T)$$



$$= \phi \circ \varphi(\alpha_G)$$

where the final equality uses Proposition 2.3.13.

COROLLARY 3.1.14. *The set* $\{\epsilon_I^J\}_{I,J\in \mathrm{Irr}(\mathcal{C})}$ *is a set of orthogonal primitive idempotents.*

Proof. Let I, J, I', J' be in $\mathrm{Irr}(\mathcal{C})$. By Theorem 3.1.13 we have

$$\mathrm{Hom}(\epsilon_I^J, \epsilon_{I'}^{J'}) = \begin{cases} \mathbb{K} & \text{if } I = I' \text{ and } J = J' \\ 0 & \text{else} \end{cases}$$

which proves the claim.

□

THEOREM 3.1.15. *Let $\mathcal{C}$ be an MTC. We have*

$$\mathrm{Hom}_{\mathcal{TC}}(X,Y) = \bigoplus_{IJ} \mathrm{Hom}_{\mathcal{TC}}(X,\epsilon_I^J) \otimes \mathrm{Hom}_{\mathcal{TC}}(\epsilon_I^J, Y),$$

in other words, the set $\{\epsilon_I^J\}_{I,J\in\mathrm{Irr}(\mathcal{C})}$ forms a complete *set of orthogonal primitive idempotents in $\mathcal{TC}$.*

Proof. We start by noting that composition gives a natural map

$$\bigoplus_{IJ} \mathrm{Hom}_{\mathcal{TC}}(X,\epsilon_I^J) \otimes \mathrm{Hom}_{\mathcal{TC}}(\epsilon_I^J, Y) \to \mathrm{Hom}_{\mathcal{TC}}(X,Y). \tag{3.6}$$

As before, let $\mathrm{hom}_{\mathcal{C}}(X,Y)$ denote the dimension of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. By Corollary 3.1.14, (3.6) is injective and therefore

$$\sum_{IJ} \mathrm{hom}_{\mathcal{TC}}(X,\epsilon_I^J)\, \mathrm{hom}_{\mathcal{TC}}(\epsilon_I^J, Y) \leq \mathrm{hom}_{\mathcal{TC}}(X,Y) \tag{3.7}$$

with equality if and only if (3.6) is an isomorphism. Furthermore, by Proposition 3.1.12 we have

$$\mathrm{hom}_{\mathcal{C}}(X,IJ) = \mathrm{hom}_{\mathcal{TC}}(X,\epsilon_J^I) \quad \text{and} \quad \mathrm{hom}_{\mathcal{C}}(IJ,Y) = \mathrm{hom}_{\mathcal{TC}}(\epsilon_J^I,Y)$$

which allows us to compute

$$\begin{aligned}
\mathrm{hom}_{\mathcal{TC}}(X,Y) &= \sum_{I^\vee} \mathrm{hom}_C(I^\vee X, YI^\vee) \\
&= \sum_{I^\vee, J} \mathrm{hom}_C(I^\vee X, J)\, \mathrm{hom}_C(J, YI^\vee) \\
&= \sum_{I,J} \mathrm{hom}_C(X, IJ)\, \mathrm{hom}_C(IJ, Y) \\
&= \sum_{IJ} \mathrm{hom}_{\mathcal{TC}}(X,\epsilon_I^J)\, \mathrm{hom}_{\mathcal{TC}}(\epsilon_I^J, Y),
\end{aligned}$$

implying that (3.6) is an isomorphism.

□

COROLLARY 3.1.16. *For $\mathcal{C}$ an MTC, the set $\{(IJ,\epsilon_I^J)^\sharp\}_{I,J\in\mathrm{Irr}(\mathcal{C})}$ forms a complete set of simples in $\mathcal{RTC}$. Furthermore, $\mathcal{RTC}$ together with the Yoneda embedding is an* *idempotent completion of $\mathcal{TC}$.*

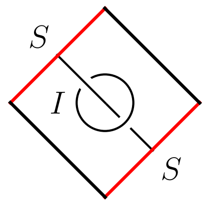
Proof. This follows directly from Theorem 3.1.15, Proposition 2.1.17 and Corollary 2.1.18. $\square$

3.1.3 | Semisimplicity of $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$

Let $\mathcal{C}$ be an MTC. Recall from Remark 3.1.5 that $\mathrm{End}_{\mathcal{TC}}(\mathbf{1}) = \mathcal{K}_{\mathbb{K}}(\mathcal{C})$ where $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$ denotes the complexified Grothendieck ring of $\mathcal{C}$. By Proposition 3.1.12 we have

$$\mathrm{Hom}_{\mathcal{TC}}(\mathbf{1}, e_I^J) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, IJ) = \delta_{J,I^\vee}\,\mathbb{K} \quad \forall I, J \in \mathrm{Irr}(\mathcal{C}).$$

Combining this with Theorem 3.1.15 and the fact that $\dim \mathcal{K}_{\mathbb{K}}(\mathcal{C}) = \#\,\mathrm{Irr}(\mathcal{C})$ and we may conclude that $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$ is a commutative semisimple algebra generated by a set of primitive orthogonal idempotents indexed by $\mathrm{Irr}(\mathcal{C})$. For $I \in \mathrm{Irr}(\mathcal{C})$ the primitive idempotent of $\mathbf{1}$ that factors through $\epsilon_I^{I^\vee}$ is given by

$$\mathbf{1}_I := \frac{1}{d(I)d(\mathcal{C})} \sum_S d(S)$$


.

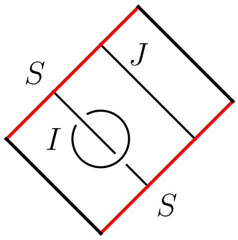
For a proof that this is indeed a primitive idempotent that factors through $\epsilon_I^{I^\vee}$ see [HK19, Corollary 5.7]. Before proceeding we make the following definition.

Definition 3.1.17. The *spectrum* of an algebra A is the set of algebra homomorphisms from A to the underlying field.

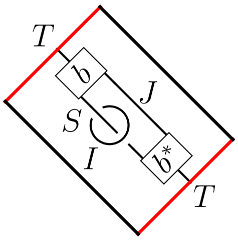
We may now compute the spectrum of $\mathcal{K}_{\mathbb{K}}(C)$ by evaluating the action of $\mathbf{1}_I$

on $[J] \in \mathcal{K}_{\mathbb{K}}(C)$ under the regular representation. We have

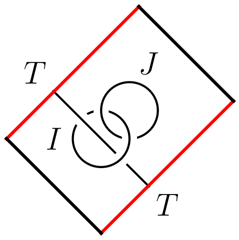
$$\mathbf{1}_I \cdot [J] = \frac{1}{d(I)d(\mathcal{C})} \sum_S d(S)$$



$$= \frac{1}{d(I)d(\mathcal{C})} \bigoplus_T \sum_{S,b} d(S)$$



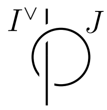
$$= \frac{1}{d(I)d(\mathcal{C})} \bigoplus_T d(T)$$



$$= \lambda_I([J])\, \mathbf{1}_I$$

where the penultimate equality uses Lemma 2.3.8 in an analogous way to the proof of Proposition 3.1.12 and $\lambda_I([J]) \in \mathbb{K}$ is defined by

$$\lambda_I([J])\, \mathrm{id}_{T^\vee} = \qquad . \tag{3.8}$$



Taking the trace of both sides of (3.8) implies that $\lambda_I([J]) = \frac{\mathcal{S}_{IJ}}{d(I)}$ where $\mathcal{S}$ is the S-matrix of $\mathcal{C}$, see (2.7). This recovers the famous fact that the S-matrix diagonalizes the fusion rules. The spectrum of $\mathcal{K}_{\mathbb{K}}(\mathcal{C})$ is therefore given by

$$\left\{ \lambda_I \colon [J] \mapsto \frac{\mathcal{S}_{IJ}}{d(I)} \quad \forall J \in \mathrm{Irr}(\mathcal{C}) \right\}_{I \in \mathrm{Irr}(\mathcal{C})} .$$

3.2 | The Category of Representations of $\mathcal{TC}$

3.2.1 | The Cycle Morphisms

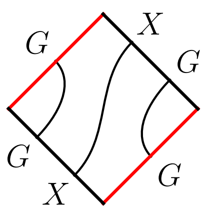
Let $\mathcal{C}$ be a spherical fusion category and, as usual, let $\mathcal{RTC}$ be the category of contravariant functors from $\mathcal{TC}$ to $\underline{\text{Vect}}$. As $\mathrm{Hom}_{\mathcal{C}}(X,Y) \leq \mathrm{Hom}_{\mathcal{TC}}(X,Y)$ such that

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y,Z) \arrow[r, hook] \arrow[d, "\circ"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_{\mathcal{TC}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{TC}}(Y,Z) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[r, hook] & \mathrm{Hom}_{\mathcal{TC}}(X,Z)
\end{tikzcd}$$

we have a canonical (covariant) functor

$$\begin{aligned} \mathcal{RTC} &\to \mathcal{RC} \\ F &\mapsto \bar{F} \end{aligned}$$

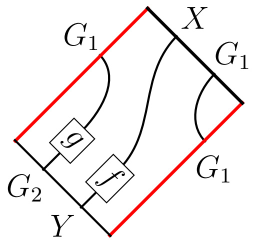
that simply restricts F to morphisms in $\mathcal{C}$. A natural question now arises: for a given object $\bar{F}$ in $\mathcal{RC}$ what additional data could be provided to specify a unique extension to an object F in $\mathcal{RTC}$? To answer this question we consider the following morphisms in $\mathcal{TC}$:

$c_{G,X} =$ 

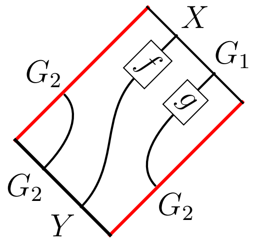
where G and X are in $\mathcal{C}$.

For f and g in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(G_1, G_2)$ respectively, we have

$$(g \otimes f) \circ c_{G_1,X} =$$



$$=$$



$$= c_{G_2,Y} \circ (f \otimes g). \tag{3.9}$$

Furthermore, $c_{G,X}$ also satisfies the following properties

- ☐ $c_{G,X}$ is an isomorphism.
- ☐ $c_{G,HX} \circ c_{H,XG} = c_{GH,X}$.

The following section proves that the data given by evaluating F on $c_{G,X}$ precisely captures how to extend $\bar{F}$ to F and therefore provides an answer to the question mentioned above.

3.2.2 | THE UNIQUE EXTENSION

Let $\mathcal{C}$ be a et F be an object in $\mathcal{RTC}$. Applying F to $c_{G,X}$ gives a collection of maps

$$\kappa_{G,X} \colon \bar{F}(GX) \to \bar{F}(XG).$$

By (3.9) κ is natural in both X and G. The additional properties of $c_{G,X}$ listed in the previous section then imply that

(i) $\kappa_{G,X}$ is an isomorphism.

(ii) $\kappa_{H,XG} \circ \kappa_{G,HX} = \kappa_{GH,X}$.

We now suppose we have an arbitrary object $\bar{F}$ in $\mathcal{RC}$ and maps

$$\kappa_{G,X} : \bar{F}(GX) \to \bar{F}(XG) \quad \forall G, X \in \mathrm{Obj}(\mathcal{C})$$

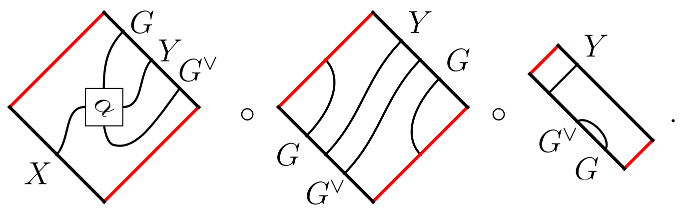
that satisfy naturality in G and X together with Properties (i) and (ii). We shall prove that there is a unique functor F in $\mathcal{RTC}$ such that

I. $F(X) = \bar{F}(X)$ for all X in $\mathcal{C}$.

II. $F(\alpha) = \bar{F}(\alpha)$ for all $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

III. $F(c_{G,X}) = \kappa_{G,X}$ for all G, X in $\mathcal{C}$.

If such a functor exists it is certainly unique as these conditions determine the functor on any morphism in $\mathcal{TC}$. Indeed for any $\alpha_G \in \mathrm{Hom}_{\mathcal{TC}}(Y, X)$ we have

$\alpha_G =$  (3.10)

The first and last terms are morphisms in $\mathcal{C}$ and therefore determined by Condition II. The middle term is $c_{G,YG^\vee}$ and is therefore determined by Condition III. The following proposition establishes existence.

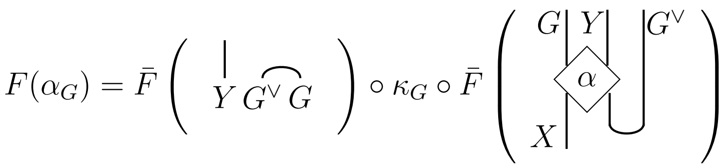
PROPOSITION 3.2.1. *There exists a unque object F in $\mathcal{RTC}$ that satisfies Conditions I, II and III.*

Proof. We first check that Conditions II and III don't contradict one another. The only case when $c_{G,X}$ is a morphism in $\mathcal{C}$ is when $G = \mathbf{1}$ and $c_{G,X} = \mathrm{id}_X$. As, by (ii) for $H = G = \mathbf{1}$, we have

$$\kappa_{\mathbf{1},X} \circ \kappa_{\mathbf{1},X} = \kappa_{\mathbf{1},X}$$

which, together with (i), implies $\kappa_{\mathbf{1},X} = \mathrm{id}_X$, Conditions II and III are equivalent in this case.

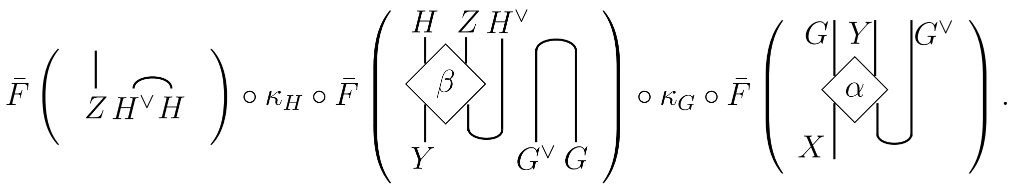
To aid legibility when the domain of a map of the form $\kappa_{G,X}$ is clear from the context we will suppress the second argument and simply write κ_G. As discussed before the proof, any F that satisfies Conditions I, II and III also satisfies



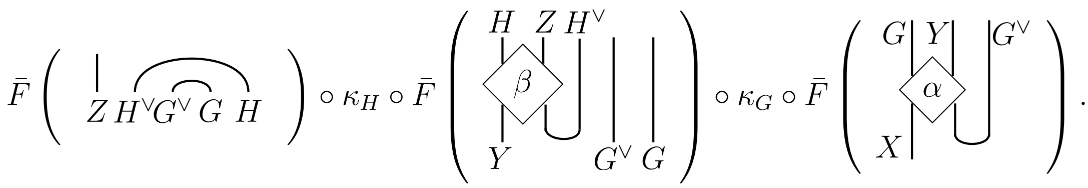
(3.11)

for any $\alpha_G \in \mathrm{Hom}_{\mathcal{TC}}(Y, X)$. We therefore only have to check that (3.11) does indeed define a functor. Let β_H be in $\mathrm{Hom}_{\mathcal{TC}}(Z, Y)$. We have

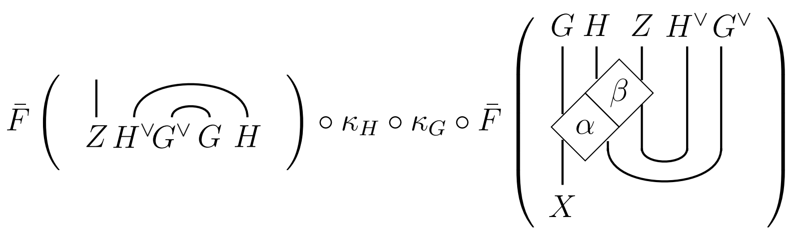
$$F(\beta_H) \circ F(\alpha_G) =$$



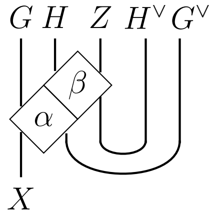
By the naturality of $\kappa_{H^\vee}$ and $\kappa_{G^\vee}$ this equation may be rearranged. The creation morphism of G in the middle term may be moved to after κ_H, giving



Then β together with the annihilation morphism of H may be moved to before $\kappa_{G^\vee}$. This yields



$=$ $\bar{F}\Bigg($ 

$\Bigg) \circ \kappa_{GH} \circ \bar{F}\Bigg($ 

$\Bigg)$

$$= F(\alpha_G \circ \beta_H)$$

where the penultimate equality uses Property (ii).

□

DEFINITION 3.2.2. We call the functor constructed in Proposition 3.2.1 the *extension of $\bar{F}$ by κ* and denote it $(\bar{F}, \kappa)$.

This new description of objects in $\mathcal{RTC}$ then also yields a new description of morphisms in $\mathcal{RTC}$ as follows.

PROPOSITION 3.2.3. *We consider $F = (\bar{F}, \kappa)$ and $F' = (\bar{F}', \kappa')$ in $\mathcal{RTC}$ then we have*

$$\mathrm{Hom}_{\mathcal{RTC}}(F, F') = \{\alpha \in \mathrm{Hom}_{\mathcal{RC}}(\bar{F}, \bar{F}') \mid \alpha_{XG} \circ \kappa_{G,X} = \kappa'_{G,X} \circ \alpha_{GX}\}.$$

Proof. Let $\alpha \in \mathrm{Hom}_{\mathcal{RC}}(\bar{F}, \bar{F}')$ be such that $\alpha_{XG} \circ \kappa_{G,X} = \kappa'_{G,X} \circ \alpha_{GX}$. As α is in $\mathrm{Hom}_{\mathcal{RC}}(\bar{F}, \bar{F}')$, α is natural with respect to all morphism in $\mathcal{C}$. Furthermore, the additional condition on α implies that it is also natural with respect to all morphisms of the form $c_{G,X}$. From (3.10) we see that any morphism in $\mathcal{TC}$ may be written as a composition of morphisms in C and morphisms of the form $c_{G,X}$. Therefore α is natural with respect to all morphisms in $\mathcal{TC}$.

□

3.3 | Equivalent Categories to $\mathcal{RTC}$

3.3.1 | Equivalence Between $\mathcal{RTC}$ and $Z(\mathcal{C})$

Definition 3.3.1. Let $\mathcal{C}$ be a monoidal category and let X be an object in $\mathcal{C}$. A *half-braiding* on X is a collection of natural isomorphisms

$$\tau_G \colon G \otimes X \to X \otimes G$$

such that

$$\tau_{GH} = (\tau_G \otimes \mathrm{id}_H) \circ (\mathrm{id}_G \otimes \tau_H)$$

for all G, H in $\mathcal{C}$.

The *centre* of $\mathcal{C}$, denoted $Z(\mathcal{C})$, is a category with objects of the form (X, τ) where X is in $\mathcal{C}$ and τ is a half braiding on X. $\mathrm{Hom}_{Z(\mathcal{C})}((X,\tau),(Y,\gamma))$ is then given by the subspace of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ defined by the condition that $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ satisfies

$$(f \otimes \mathrm{id}_G) \circ \tau_G = \gamma_G \circ (\mathrm{id}_G \otimes f)$$

for all G in $\mathcal{C}$. This category is monoidal [EGNO15, Section 7.13] with tensor product $(X,\tau) \otimes (Y,\gamma) = (X \otimes Y, \iota)$ where $\iota_G = (\mathrm{id}_X \otimes \gamma_G) \circ (\tau_G \otimes \mathrm{id}_Y)$. In fact $Z(\mathcal{C})$ also admits a natural braiding [Kas98, Theorem XIII.4.2.] given by

$$\sigma_{(X,\tau),(Y,\gamma)} = \gamma_X. \tag{3.12}$$

Now let $\mathcal{C}$ be a spherical fusion category. Then $\mathcal{C}$ satisfies the conditions of Proposition 2.1.15 and so the Yoneda embedding gives an equivalence between $\mathcal{C}$ and $\mathcal{RC}$. This induces an equivalence

$$\begin{aligned} Z(\mathcal{C}) &\to Z(\mathcal{RC}) \\ (X,\tau) &\mapsto (X^\sharp, \tau^\sharp). \end{aligned}$$

Here $\tau^\sharp$ is a natural isomorphism from $(-\otimes X)^\sharp$ to $(X \otimes -)^\sharp$ such that

$$\tau^\sharp_{GH} = (\tau_G \otimes \mathrm{id}_H)^\sharp \circ (\mathrm{id}_G \otimes \tau_H)^\sharp.$$

By Lemma 2.2.10 this is equivalent to an isomorphism

$$\kappa_{G^\vee,Z} \colon X^\sharp(G^\vee Z) \to X^\sharp(ZG^\vee)$$

that is natural in both Z and $G^\vee$ and satisfies

$$\kappa_{(HG)^\vee,Z} = \kappa_{H^\vee,ZG^\vee} \circ \kappa_{G^\vee,H^\vee Z}$$

or, more simply,

$$\kappa_{GH,Z} = \kappa_{H,ZG} \circ \kappa_{G,HZ}.$$

Therefore, up to isomorphism, an object in $Z(\mathcal{RC})$ may simply be thought of as an object in $\mathcal{RC}$ together with maps $\kappa_{G,Z}$ as above, i.e. an object in $\mathcal{RTC}$ (cf. Section 3.2.2). Furthermore, we recall that a morphism in $Z(\mathcal{C})$ between (X,τ) and (Y,τ') is a map $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that

$$(f \otimes \mathrm{id}_G) \circ \tau_G = \tau'_G \circ (\mathrm{id}_G \otimes f).$$

Applying the Yoneda embedding we get

$$\Big((f \otimes \mathrm{id}_G)^\sharp \circ \tau^\sharp_G\Big)_Z = \Big((\tau')^\sharp_G \circ (\mathrm{id}_G \otimes f)^\sharp\Big)_Z$$

which is equivalent to

$$f^\sharp_{ZG^\vee} \circ \kappa_{G^\vee,Z} = \kappa'_{G^\vee,Z} \circ f^\sharp_{G^\vee Z}$$

which is precisely the condition that $f^\sharp$ is a morphism from $(X^\sharp, \kappa_{G^\vee,Z})$ to $(Y^\sharp, \kappa'_{G^\vee,Z})$ in $\mathcal{RTC}$. Therefore $\mathcal{RTC}$ and $Z(\mathcal{C})$ are equivalent. This equivalence has been previously proved using the tube *algebra* in [PSV18, Proposition 3.14].

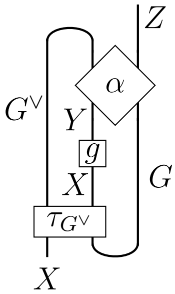
To see this equivalence more explicitly let (X,τ) be in $Z(\mathcal{C})$. The corresponding object F in $\mathcal{RTC}$ is given on objects by

$$F(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

and on morphisms by

$$F(\alpha_G)\colon\ \mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, X)$$

$$g \mapsto$$



.

where $\alpha_G \in \mathrm{Hom}_{\mathcal{TC}}(Z, Y)$.

REMARK 3.3.2. The equivalence between the centre of a category and the idempotent completion of the corresponding tube category probably pre-dates even [PSV18, Proposition 3.14]. In [Jr11], Kirillov defines a category $\hat{\mathcal{C}}(S)$ for a spherical fusion category $\mathcal{A}$ and an oriented 1-dimensional manifold S. It seems very likely that, when S is a circle, $\hat{\mathcal{C}}(S)$ will coincide with the tube category of $\mathcal{A}$. In particular, Kirillov proves that the idempotent completion of $\hat{\mathcal{C}}(S)$ is $Z(\mathcal{A})$ [Jr11, Theorem 6.4].

3.3.2 | EQUIVALENCE BETWEEN $Z(\mathcal{C})$ AND $\mathcal{C} \boxtimes \overline{\mathcal{C}}$

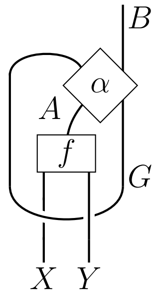
Let $\mathcal{C}$ be a PTC. There is a well-known (covariant) braided monoidal functor

$$\begin{aligned} \Phi\colon\ & \mathcal{C} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{C}) \\ & X \boxtimes Y \mapsto (XY, (\mathrm{id}_X \otimes \bar{\sigma_Y}) \circ (\sigma_X \otimes \mathrm{id}_Y)) \end{aligned}$$

where $\boxtimes$ denotes the Deligne tensor product and $\overline{\mathcal{C}}$ is obtained by equipping $\mathcal{C}$ with the opposite braiding. It is also know that this functor is an equivalence if and only if $\mathcal{C}$ is modular (see [Müg03] or [EGNO15, Proposition 8.20.12]).

Under the equivalence described in Section 3.3.1, $\Phi(X \boxtimes Y)$ corresponds to a object in $\mathcal{RTC}$ given by $\Phi(X \boxtimes Y)(Z) = \mathrm{Hom}_{\mathcal{C}}(Z, XY)$ for Z in $\mathcal{C}$ and

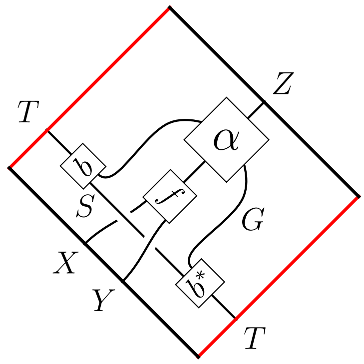
$$\Phi(X \boxtimes Y)(\alpha_G)\colon \ \mathrm{Hom}_{\mathcal{C}}(A, XY) \to \mathrm{Hom}_{\mathcal{C}}(B, XY)$$

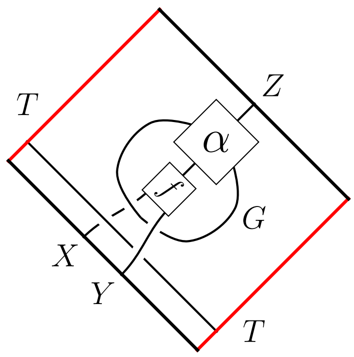
$f \mapsto$ 

for $\alpha_G \in \mathrm{Hom}_{\mathcal{TC}}(B, A)$. The following theorem establishes a relationship between this functor and the idempotents ϵ_X^Y introduced in Section 3.1.2.

THEOREM 3.3.3. *Let X and Y be in $\mathrm{Irr}(\mathcal{C})$. Then $\Phi(X \boxtimes Y) = (XY, \epsilon_X^Y)^\sharp$.*

Proof. We have show that the isomorphism $\phi_Z\colon \ \mathrm{Hom}_{\mathcal{C}}(Z, XY) \to \mathrm{Hom}_{\mathcal{TC}}(Z, e_X^Y)$ from the proof of Proposition 3.1.12 is natural in this context. For $f \in \mathrm{Hom}_{\mathcal{C}}(A, XY)$ and $\alpha_G \in \mathrm{Hom}_{\mathcal{TC}}(B, A)$, we have

$$(XY, \epsilon_X^Y)^\sharp(\alpha_G) \circ \phi_A(f) = \frac{1}{d(\mathcal{C})} \bigoplus_T \sum_{S,b} d(S)$$ 

$$= \frac{1}{d(\mathcal{C})} \bigoplus_T d(T)$$ 

$$= \phi_B \circ \Phi(X \boxtimes Y)(\alpha_G)(f)$$

where the penultimate equality uses Lemma 2.3.8 in an analogous way to the proof of Proposition 3.1.12.

□

The results of Section 3.1.2 many now be interpreted as a graphical proof of a result due to Müger [Müg03] mentioned at the start of this section.

COROLLARY 3.3.4. *When $\mathcal{C}$ is modular Φ is an equivalence.*

Proof. Theorem 3.1.15 proves that Φ is essentially surjective and Theorem 3.1.13 proves that Φ is fully faithful. □

3.3.3 | GRAPHICAL DESCRIPTION OF THE TENSOR PRODUCT AND BRAIDING

Let $\mathcal{C}$ be a PTC. $\mathcal{RTC}$ admits a monoidal product and braiding induced by the corresponding structure on $Z(\mathcal{C})$ (cf. Section 3.3.1). It is also rigid and inherits a pivotal structure from $\mathcal{C}$. The aim of this section is to give a graphical description of these structures in terms of the idempotents ϵ_X^Y.

PROPOSITION 3.3.5. *For X, Y objects in $\mathcal{C}$ let ϵ_X^Y be as in Section 3.1.2. Then we have*

$$(XY, \epsilon_X^Y)^\sharp \otimes (AB, \epsilon_A^B)^\sharp = (XAYB, \epsilon_{XA}^{YB})^\sharp = (XYAB, \epsilon_X^Y \otimes_{\mathcal{TC}} \epsilon_A^B)^\sharp.$$

Furthermore, if $\mathcal{C}$ is modular, then for $\alpha \in \mathrm{Hom}_{\mathcal{TC}}(\epsilon_X^Y, \epsilon_A^B)$ and $\beta \in \mathrm{Hom}_{\mathcal{TC}}(\epsilon_{X'}^{Y'}, \epsilon_{A'}^{B'})$, we have

$$\alpha \otimes \beta = \alpha \otimes_{\mathcal{TC}} \beta$$

where $\otimes_{\mathcal{TC}}$ is the associative product given by Definition 3.1.7.

Proof. By Theorem 3.3.3 we have that

$$\begin{aligned}(XY, \epsilon_X^Y)^\sharp \otimes (AB, \epsilon_A^B)^\sharp &= \Phi(X \boxtimes Y) \otimes \Phi(A \boxtimes B)\\ &= \Phi(XA \boxtimes YB)\\ &= (XAYB, \epsilon_{XA}^{YB})^\sharp\end{aligned}$$

where Φ is as defined in Section 3.3.2. As described in Remark 3.1.11, we then have

$$(XAYB, \epsilon_{XA}^{YB})^\sharp = (XYAB, \epsilon_X^Y \otimes_{\mathcal{TC}} \epsilon_A^B)^\sharp.$$

where the natural isomorphism is given by the braiding. This proves the first half of the proposition.

Let f, f', g, g' be in $\mathrm{Hom}_{\mathcal{C}}(X, A)$, $\mathrm{Hom}_{\mathcal{C}}(X', A')$, $\mathrm{Hom}_{\mathcal{C}}(Y, B)$ and $\mathrm{Hom}_{\mathcal{C}}(Y', B')$ respectively and let α and β be given by

$$\alpha = \Phi(f \boxtimes g) \quad \text{and} \quad \beta = \Phi(f' \boxtimes g'). \tag{3.13}$$

Then

$$\begin{aligned}\alpha \otimes \beta &= \Phi((f \boxtimes g) \otimes (f \boxtimes g))\\ &= \Phi((f \otimes f') \boxtimes (g \otimes g'))\\ &= \alpha \otimes_{\mathcal{TC}} \beta.\end{aligned}$$

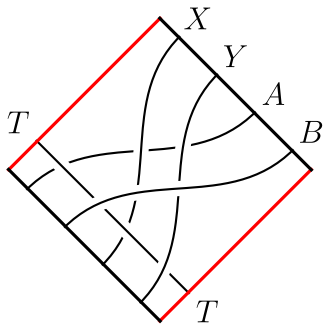
as desired.

We now assume that $\mathcal{C}$ is modular. By Corollary 3.3.4, Φ is fully faithful. Therefore any morphism in $\mathcal{TC}$ may be written as a sum of morphisms of the form (3.13). This implies $\alpha \otimes \beta = \alpha \otimes_{\mathcal{TC}} \beta$ for arbitrary α and β.

□

The braiding between $(XY, \epsilon_X^Y)^\sharp$ and $(AB, \epsilon_A^B)^\sharp$ is then given by the following morphism, cf. (3.12),

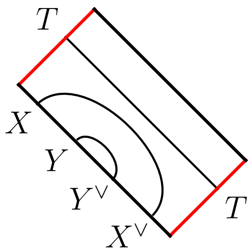
$$\sigma^{Y,B}_{X,A} = \frac{1}{d(\mathcal{C})} \bigoplus d(T)$$



$$\begin{aligned} &\in \mathrm{Hom}_{\mathcal{TC}}(\epsilon^Y_X \otimes_{\mathcal{TC}} \epsilon^B_A, \epsilon^B_A \otimes_{\mathcal{TC}} \epsilon^Y_X) \\ &= \mathrm{Hom}_{\mathcal{RTC}}((XYAB, \epsilon^Y_X \otimes_{\mathcal{TC}} \epsilon^B_A)^\sharp, (ABXY, \epsilon^B_A \otimes_{\mathcal{TC}} \epsilon^Y_X)^\sharp) \end{aligned}$$

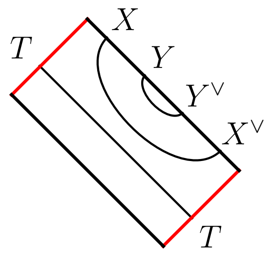
and the creation and annihilation morphisms for $(XY, \epsilon^Y_X)^\sharp$ and $(XY, \epsilon^Y_X)^\sharp$ are given by

$$\frac{1}{d(\mathcal{C})} \bigoplus d(T)$$



and

$$\frac{1}{d(\mathcal{C})} \bigoplus d(T)$$



respectively. Note the tensor identity in $\mathcal{RTC}$ is $(\mathbf{1}, \epsilon^{\mathbf{1}}_{\mathbf{1}})^\sharp$ and *not* $\mathrm{Hom}_{\mathcal{TC}}(-, \mathbf{1})$.

CHAPTER 4

THE TRACE OF A MONOIDAL FUNCTOR

The principal aim of this chapter is to introduce and analyse a construction that associates a representation of the tube category to a pivotal monoidal functor on a spherical fusion category. When the category modular, this representation possesses several interesting properties related to the notion of a modular invariant.

Section 4.1 starts by defining, for a pivotal monoidal functor $\mathcal{M}$ on a spherical fusion category $\mathcal{C}$, a representation of $\mathcal{TC}$. This is achieved by first taking the trace of $\mathcal{M}$ and obtaining a functor on $\mathcal{C}$. It is then explained how the trace of $\mathcal{M}$ possesses a natural extension to a functor on $\mathcal{TC}$, which we denote $\mathcal{TM}$. Next, we assume $\mathcal{C}$ is pre-modular and provide a more explicit characterisation of the Hom-space between the Yoneda embedding of ϵ_X^Y and $\mathcal{TM}$. Finally, for F an arbitrary functor on $\mathcal{TC}$, we introduce the integer $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$-matrix $Z(F)$ whose (I, J)-entry is given by the dimension the Hom-space between the Yoneda embedding of ϵ_I^J and $\mathcal{TM}$. When $\mathcal{C}$ is modular the entries of $Z(F)$ are given by the multiplicities of simple objects in F.

The primary goal of Section 4.2 is to prove that $\mathcal{TM}$ is T-invariant, i.e. that $Z(\mathcal{TM})$ commutes with the T-matrix of $\mathcal{C}$. When $\mathcal{C}$ is modular, T-invariance may be characterised as requiring that certain distinguished morphisms in $\mathcal{TC}$ are mapped to the identity. The bulk of this section is therefore dedicated to proving this result with T-invariance of $\mathcal{TM}$ following as a corollary.

Section 4.3 first shows that taking $\mathcal{M} = \mathrm{id}_{\mathcal{C}}$ gives a simple counter-example to the hypothesis that $\mathcal{TM}$ is S-invariant, i.e. that $Z(\mathcal{TM})$ commutes with the S-matrix, in general. However, a result of Kong and Runkel [KR09, Theorem 3.4] implies that, when $\mathcal{C}$ is modular, a haploid, symmetric, commutative Frobenius algebra in $\mathcal{RTC}$ is S-invariant if and only if its dimension is equal to the dimension of $\mathcal{C}$. As this condition on the dimension is a necessary requirement for S-invariance, proving that an object in $\mathcal{RTC}$ is such a Frobenius algebra strictly weakens the condition that it be S-invariant. Section 4.3 proceeds to prove that $\mathcal{TM}$ may be equipped with the structure of a commutative algebra. Let $\mathcal{D}$ be the target category for the functor $\mathcal{M}$. Under the assumption that the idempotent completion of $\mathcal{D}$ is a multifusion category and that $\mathcal{M}$ is indecomposable, it is shown that $\mathcal{TM}$ is a haploid, symmetric, commutative Frobenius algebra.

Finally, Section 4.4 relates the $\mathcal{TM}$ construction to a technique known as α-induction. This technique, first developed by Böckenhauer and Evans [BE98], may be thought of as a procedure that accepts certain module categories over $\mathcal{C}$ and produces a modular invariant. It is shown that, under the assumption that the module category $\mathcal{M}$ induces a pivotal structure on its full image, the $\mathcal{TM}$ construction may be applied and the matrix $Z(\mathcal{TM})$ coincides with the matrix produced by α-induction. Furthermore, under the additional assumption that $\mathcal{M}$ be indecomposable, the results of Section 4.3 imply that $\mathcal{TM}$ is a haploid, symmetric, commutative Frobenius algebra.

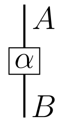
4.1 | Introduction to $\mathcal{TM}$

4.1.1 | Extension to the Tube Category

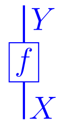
Let $\mathcal{C}$ be spherical fusion category, let $\mathcal{D}$ be a pivotal monoidal category (see Section 2.2) and let

$$\mathcal{M} \colon \mathcal{C} \to \mathcal{D}$$

be a pivotal monoidal functor. When doing graphical calculus in $\mathcal{D}$ we use blue to depict the image of objects and morphisms in $\mathcal{C}$ under $\mathcal{M}$. For example a morphism $\alpha \in \mathrm{Hom}_{\mathcal{D}}(A, B)$ is depicted in the normal way,



whereas, for $\beta \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we depict $\mathcal{M}(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(Y), \mathcal{M}(X))$ as



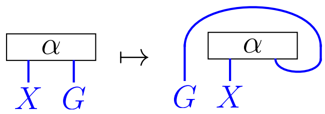
.

Composing $\mathcal{M}$ with the *trace* functor gives the following object in $\mathcal{RC}$

$$\begin{aligned} \mathrm{Tr}\,\mathcal{M} \colon \mathcal{C} &\to \underline{\mathrm{Vect}} \\ X &\mapsto \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(X)). \end{aligned}$$

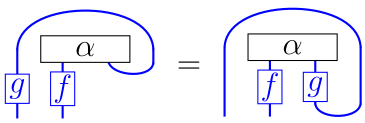
For X and G in $\mathcal{C}$ we consider the isomorphism

$$\kappa_{G,X} : \mathrm{Tr}\,\mathcal{M}(XG) \to \mathrm{Tr}\,\mathcal{M}(GX)$$



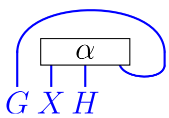
.

As, for f and g morphisms in $\mathcal{C}$,

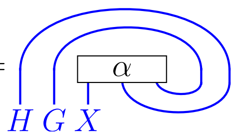


we have that $\kappa_{G,X}$ is natural in both G and X. Furthermore, we have

$\kappa_{G,HX} \circ \kappa_{H,XG} = \kappa_{G,HX}$ (



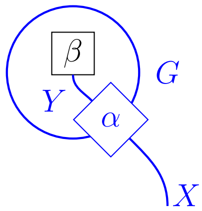
) =



$= \kappa_{GH,X}$

and $\kappa_{\mathbf{1},X} = \mathrm{id}_{\bar{F}(X)}$. We can therefore apply Proposition 3.2.1 to extend $\mathrm{Tr}\,\mathcal{M}$ to a functor on $\mathcal{TC}$. We denote this extension $\mathcal{TM}$, or, to use the notation of Section 3.2.2, $\mathcal{TM} := (\mathrm{Tr}\,\mathcal{M}, \kappa)$. For a more concrete description of $\mathcal{TM}$ we consider $\alpha_G \in \mathrm{Hom}_{\mathcal{TC}}(X, Y)$. Then we have

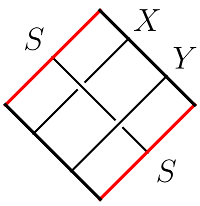
$$\mathcal{TM}(\alpha_G)\colon\ \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(Y)) \to \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(X))$$

$\beta \mapsto$ 

.

4.1.2 | Hom-Spaces Between $(XY, \epsilon_X^Y)^\sharp$ and $\mathcal{TM}$

Let $\mathcal{C}$ be a pre-modular tensor category (PTC). We recall from Section 3.1.2 the following idempotents in $\mathcal{TC}$:

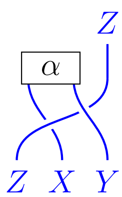
$$\epsilon_X^Y = \frac{1}{d(\mathcal{C})} \bigoplus_S d(S)$$ 

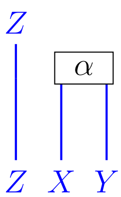
$\in \mathrm{End}_{\mathcal{TC}}(XY)$

where X and Y are in $\mathcal{C}$. For an object F in $\mathcal{RTC}$ we consider the Hom-space

$$F_X^Y := \mathrm{Hom}_{\mathcal{RTC}}((XY, \epsilon_X^Y)^\sharp, F) = \{\alpha \in F(XY) \mid F(\epsilon_X^Y)(\alpha) = \alpha\}.$$

Proposition 4.1.1. *$\mathcal{TM}_X^Y$ is given by the subspace of $\mathcal{TM}(XY) = \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(XY))$ defined by the condition that $\alpha \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(XY))$ satisfies*



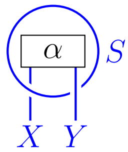
$=$


. (4.1)

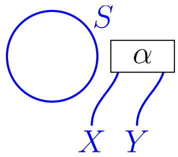
for all Z in $\mathcal{C}$.

Proof. Evaluating $\mathcal{TM}$ on ϵ_X^Y gives the map

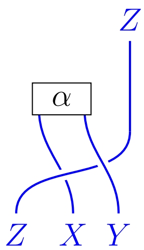
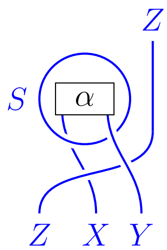
$$\mathcal{TM}(\epsilon_X^Y)\colon\ \mathrm{Hom}_{\mathcal{D}}(S, \mathcal{M}(XY)) \to \mathrm{Hom}_{\mathcal{D}}(S, \mathcal{M}(XY))$$

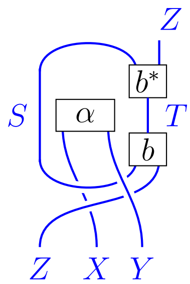
$$\alpha \mapsto \frac{1}{d(\mathcal{C})} \sum_S d(S)$$  .

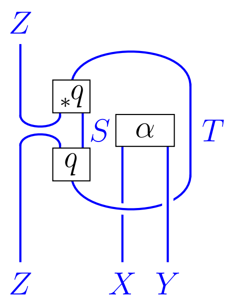
Therefore, if α satisfies (4.1), we have

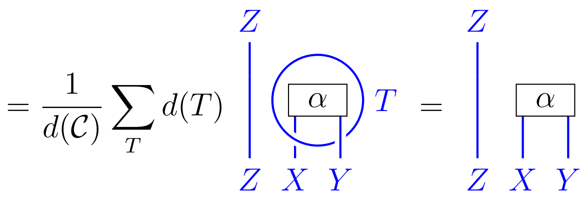
$$\mathcal{TM}(\epsilon_X^Y)(\alpha) = \frac{1}{d(\mathcal{C})} \sum_S d(S)$$  $= \alpha.$

Furthermore, for $\alpha \in \mathcal{TM}_X^Y$, we have

 $$= \frac{1}{d(\mathcal{C})} \sum_S d(S)$$ 

$$= \frac{1}{d(\mathcal{C})} \sum_{S,T,b} d(S)$$ 

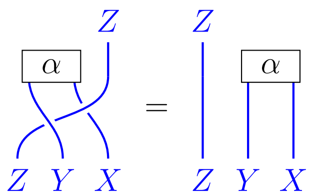
$$= \frac{1}{d(\mathcal{C})} \sum_{S,T,b} d(S)$$ 



where, to make certain string manipulations clearer, we have chosen to write b and b^* upside-down instead of writing $b^\vee$ and $(b^*)^\vee$.

□

REMARK 4.1.2. We recall from Remark 3.1.10 that $(XY, \epsilon_X^Y)^\sharp = (YX, \tilde{\epsilon}_X^Y)^\sharp$ where $\tilde{\epsilon}_X^Y$ is given by (3.5). Therefore $\mathcal{TM}_X^Y$ may also be identified with the subspace of $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(YX))$ defined by the condition that $\alpha \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(YX))$ satisfies



for all Z in $\mathcal{C}$.

DEFINITION 4.1.3. For any F in $\mathcal{RTC}$ one may consider the $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$ integer matrix.

$$Z(F) := (\dim F_I^J)_{I,J \in \mathrm{Irr}(\mathcal{C})}.$$

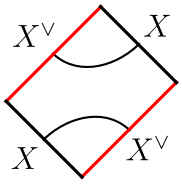
REMARK 4.1.4. We recall that, if $\mathcal{C}$ is an MTC, then by Corollary 2.1.18 the set $\{(IJ, \epsilon_I^J)^\sharp\}_{I,J\in\mathrm{Irr}(\mathcal{C})}$ forms a complete set of simples in $\mathcal{RTC}$. Therefore an entry of $Z(F)$ simply gives the multiplicity of the corresponding simple object in F.

4.2 | T-INVARIANCE

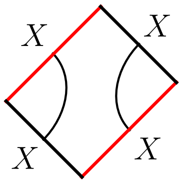
4.2.1 | THE TWIST MORPHISMS

DEFINITION 4.2.1. Let $\mathcal{C}$ be a PTC and let $\mathcal{T}$ be the T-matrix of $\mathcal{C}$ as defined by (2.7). We call an object F in $\mathcal{RTC}$ *T-invariant* if $Z(F)$ commutes with $\mathcal{T}$.

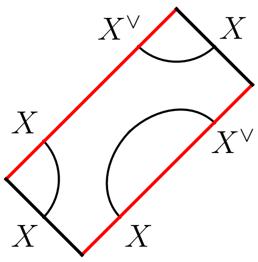
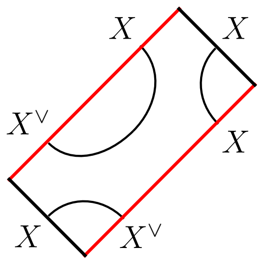
The principal goal of this section is to give a graphical characterisation of T-invariance when $\mathcal{C}$ is modular. We consider the following two automorphisms of X in $\mathcal{TC}$,

$t_X :=$  .

and

$\tilde{t}_X :=$  .

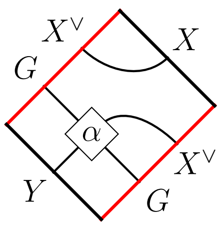
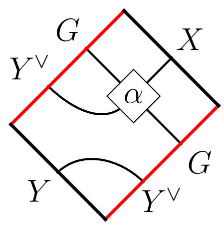
As should be expected, t_X and $\tilde{t}_X$ are inverse to one another. Indeed we have

$\tilde{t}_X \circ t_X =$  $= \mathrm{id}_X =$  $= t_X \circ \tilde{t}_X$

LEMMA 4.2.2. *For all* $\alpha \in \mathrm{Hom}_{\mathcal{TC}}(X, Y)$ *we have,*

$$\alpha \circ t_X = t_Y \circ \alpha.$$

Proof. W.o.l.g. let α_G be in $\mathrm{Hom}_{\mathcal{TC}}(X, Y)$. We have

$\alpha_G \circ t_X =$  $=$  $= t_X \circ \alpha_G.$

as desired. $\square$

4.2.2 | Characterising T-invariance in an MTC

By Theorem 3.1.15, if $\mathcal{C}$ is an MTC then ϵ_I^J is a primitive idempotent. In particular we have $\mathrm{End}_{\mathcal{TC}}(\epsilon_I^J) = \mathbb{K}$. However, by Lemma 4.2.2, we have

$$\epsilon_I^J \circ t_{IJ} \circ \epsilon_I^J = \epsilon_I^J \circ \epsilon_I^J \circ t_{IJ} = \epsilon_I^J \circ t_{IJ}$$

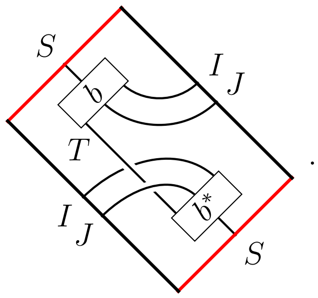
so $\epsilon_I^J \circ t_{IJ} \in \mathrm{End}_{\mathcal{TC}}(\epsilon_I^J)$. Therefore $\epsilon_I^J \circ t_{IJ} = \lambda \epsilon_I^J$ for some $\lambda \in \mathbb{K}$. This turns out to also be true in the case when $\mathcal{C}$ is a PTC.

Proposition 4.2.3. *Let $\mathcal{C}$ be a PTC and let I, J be in $\mathrm{Irr}(\mathcal{C})$. Then*

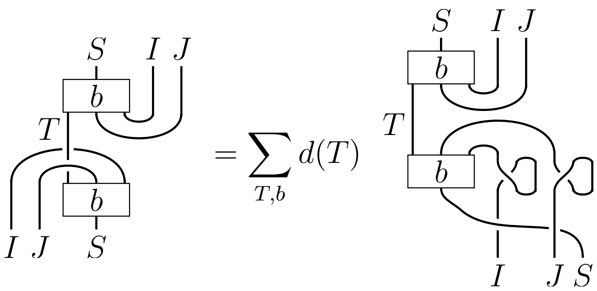
$$\epsilon_I^J \circ t_{IJ} = \frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}} \epsilon_I^J$$

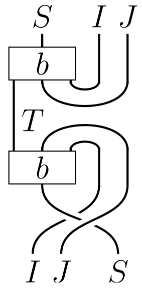
where $\mathcal{T}$ is the T-matrix of $\mathcal{C}$, see (2.7).

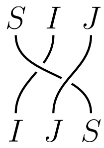
Proof. We have

$$\epsilon_I^J \circ t_{IJ} = \bigoplus_S \sum_{T,b} d(T)$$ 

Therefore the S-summand of $\epsilon_I^J \circ t_{IJ}$ is given by

$$\sum_{T,b} d(T) \quad = \sum_{T,b} d(T)$$ 

$$= \frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}} \sum_{T,b} d(T)$$ 

$$= \frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}} d(S)$$ 

where the final equality is due to Lemma 2.3.8. As this is exactly the S-summand of $\frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}}\epsilon_I^J$ we are done.

□

We may now prove the main result of this section.

THEOREM 4.2.4. *Let $\mathcal{C}$ be an MTC and let F be an object in $\mathcal{RTC}$. F is T-invariant if and only if $F(t_X) = \mathrm{id}_X$ for all X in $\mathcal{C}$.*

Proof. As $\mathcal{C}$ is an MTC the $(IJ, \epsilon_I^J)^\sharp$ form a complete set of simples. We can therefore decompose F as

$$F = \bigoplus_{IJ} F_I^J \cdot (IJ, \epsilon_I^J)^\sharp.$$

Evaluating this on t_X gives

$$F(t_X) = \bigoplus_{IJ} \mathrm{id}_{F_I^J} \otimes (IJ, \epsilon_I^J)^\sharp(t_X).$$

By Lemma 4.2.2 and Proposition 4.2.3 we have, for $\alpha \in \mathrm{Hom}_{\mathcal{TC}}(X, \epsilon_I^J)$,

$$(IJ, \epsilon_I^J)^\sharp(t_X)(\alpha) = \epsilon_I^J \circ \alpha \circ t_X = \epsilon_I^J \circ t_{IJ} \circ \alpha = \frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}}\epsilon_I^J \circ \alpha = \frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}}\alpha.$$

Therefore

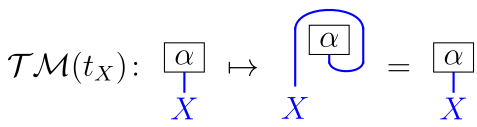
$$\bigoplus_{IJ} \mathrm{id}_{F_I^J} \otimes (IJ, \epsilon_I^J)^\sharp(t_X) = \bigoplus_{IJ} \frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}} \mathrm{id}_{F_I^J \otimes (IJ, \epsilon_I^J)^\sharp(X)} \,.$$

This is equal to $\bigoplus\limits_{IJ} \mathrm{id}_{F_I^J \otimes (IJ, \epsilon_I^J)^\sharp(X)} = \mathrm{id}_X$ if and only if $F_I^J \neq 0$ implies $\frac{\mathcal{T}_{II}}{\mathcal{T}_{JJ}} = 1$. As $\mathcal{T}$ is diagonal that is precisely the condition that $Z(F)$ commutes with $\mathcal{T}$.

$\square$

COROLLARY 4.2.5. *If $\mathcal{C}$ is an MTC then $\mathcal{TM}$ is T-invariant.*

Proof. For $\alpha \in \mathcal{TM}(X) = \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(X))$ we have



as $\mathcal{M}$ is pivotal. Therefore Theorem 4.2.4 applies, and $\mathcal{TM}$ is T-invariant.

$\square$

REMARK 4.2.6. Theorem 4.2.4 may be rephrased as stating that a functor being T-invariant is equivalent to that functor being well defined on the category obtained by quotienting $\mathcal{TC}$ by the two sided ideal generated by $t_X - \mathrm{id}_X$. The resulting quotient category may be thought of as the true cylindrical analogue to $\mathcal{C}$ as the quotient forgets the choice of a distinguished point on S^1 that the definition of $\mathcal{TC}$ implicitly makes (cf. Section 3.1.1).

4.3 | S-INVARIANCE AND FROBENIUS ALGEBRAS

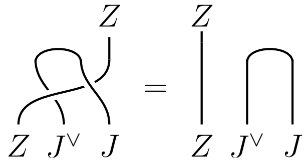
DEFINITION 4.3.1. Let $\mathcal{C}$ be a PTC and let $\mathcal{S}$ be the S-matrix of $\mathcal{C}$ as defined by (2.7). We call an object F in $\mathcal{RTC}$ *S-invariant* if $Z(F)$ commutes with $\mathcal{S}$.

We start with an example which illustrates that, even when $\mathcal{C}$ is modular, $\mathcal{TM}$ is *not* necessarily S-invariant.

EXAMPLE 4.3.2. Let $\mathcal{C}$ be an MTC and let $\mathcal{M} = \mathrm{id}_{\mathcal{C}}$. Then

$$Z(\mathcal{TM})_{IJ} \leq \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, IJ) = \delta_{I,J^\vee} \langle \mathrm{cr}_{J^\vee} \rangle_{\mathbb{K}} \,.$$

We now suppose $\delta_{I,J^\vee} = 1$. By Proposition 4.1.1, $Z(\mathcal{TM})_{J^\vee J}$ is non-trivial if and only if



for all Z in $\mathcal{C}$. Post-composing this equality with $\mathrm{id}_Z \otimes \mathrm{an}_J$ and taking the trace implies $\mathcal{S}_{ZJ} = d(Z)d(J)$ for all Z in $\mathcal{C}$. Therefore the J-th column in $\mathcal{S}$ is proportional to the **1**-th column. As $\mathcal{C}$ is modular this implies $J = \mathbf{1}$. In summary, we have

$$\mathcal{TM}_{IJ} = \begin{cases} 1 & \text{if } I = J = \mathbf{1} \\ 0 & \text{else.} \end{cases}$$

Conjugating this matrix with $\mathcal{S}$ and using Remark 2.2.25 gives us

$$\left(\mathcal{S}\,\mathcal{TM}\,\mathcal{S}^{-1}\right)_{\mathbf{11}} = \frac{1}{d(\mathcal{C})}$$

implying that $\mathcal{S}$-invariance will fail whenever $d(\mathcal{C}) \neq 1$.

However, when $\mathcal{C}$ is an MTC, we do have the following helpful theorem from [KR09].

THEOREM 4.3.3 (Theorem 3.4, [KR09]). *Let A be a haploid, symmetric, commutative Frobenius algebra (see Definition 4.3.7, 4.3.10, 4.3.11 and 4.3.12) in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. Then the* $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$*-matrix with entries* $\hom(I \boxtimes J, A)$ *commutes with the S-matrix of $\mathcal{C}$ (where, as before,* $\hom$ *denotes the dimension of the relevant* Hom*-space) if and only if*

$$d(A) = d(C). \tag{4.2}$$

REMARK 4.3.4. We note that

$$\begin{aligned} d(A) &= \sum_{IJ} \hom_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(I \boxtimes J, A) d(I \boxtimes J) \\ &= \sum_{IJ} \hom_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(I \boxtimes J, A) d(I) d(J) \\ &= \sum_{IJ} \mathcal{S}_{\mathbf{1},I} \hom_{\mathcal{C}\boxtimes\overline{\mathcal{C}}}(I \boxtimes J, A) \mathcal{S}_{J,\mathbf{1}}. \end{aligned}$$

As $\mathcal{S}_{I,J} = d(\mathcal{C})\mathcal{S}^{-1}_{I,J^\vee}$ (c.f Remark 2.2.25) Condition 4.2 is precisely the condition that the matrix with entries $\hom(I \boxtimes J, A)$ commutes with the S-matrix evaluated at $(\mathbf{1}, \mathbf{1})$ for an arbitrary object A in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. Condition (4.2) is therefore certainly necessary, the content of the theorem is that, when A is a haploid, symmetric, commutative Frobenius algebra, it is also sufficient.

REMARK 4.3.5. [KR09, Theorem 3.4] actually proves that when A is a haploid, symmetric, commutative Frobenius algebra (4.2) implies an equality that is strictly stronger than the result stated here. In particular, [KR09, Theorem 3.4] proves that A will be a *modular invariant algebra*. This notion is defined and motivated in [Kon08, Section 6].

REMARK 4.3.6. As explained in the proof of [KR09, Theorem 3.4], there exists an MTC (the category of of local A-modules) whose dimension is given by $\frac{d(\mathcal{C})^2}{d(A)^2}$. Combining this with the fact that any MTC over the complex numbers has dimension at least 1 [ENO05, Theorem 2.3.] tells us that, in the case when $\mathbb{K} = \mathbb{C}$, the dimention of A cannot exceed $d(\mathcal{C})$.

We recall that, when $\mathcal{C}$ is modular, $\Phi\colon\ \mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{RTC}$ is an equivalence and $I \boxtimes J \mapsto (IJ, \epsilon^J_I)^\sharp$ (see Theorem 3.3.3). Therefore, for F in $\mathcal{RTC}$, the $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$ matrix with entries $\hom_{\mathcal{RTC}}(\Phi(I \boxtimes J), F)$ is precisely $Z(F)$. The goal of this section is to prove that $\mathcal{TM}$ is a commutative algebra in $\mathcal{RTC}$, and then, under a further condition on $\mathcal{M}$, to show that it is also a haploid, symmetric, commutative Frobenius algebra.

4.3.1 | PRELIMINARIES ON FROBENIUS ALGEBRAS

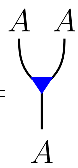
Let $\mathcal{B}$ be any monoidal category. We start by developing some preliminary facts about Frobenius algebras in $\mathcal{B}$.

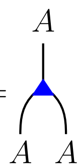
DEFINITION 4.3.7. A *Frobenius algebra* A is an algebra and a coalgebra such that

$$(\mathrm{id}_A \otimes \nabla) \circ (\Delta \otimes \mathrm{id}_A) = \Delta \circ \nabla = (\nabla \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta) \tag{4.3}$$

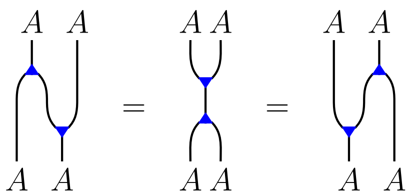
where ∇ is the product and Δ is the coproduct.

REMARK 4.3.8. Using the graphical notation

$\nabla =$ 

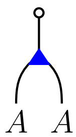
and $\Delta =$ 

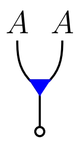
we can rewrite Condition (4.3) as



. (4.4)

An important property of Frobenius algebras is that they naturally carry a self-dual structure. Indeed, it is simple to check that the maps



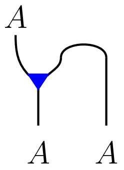
and


(4.5)

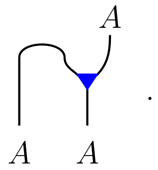
where  denotes the *unit* and  denotes the *counit*, are self-dualizing maps on A.

Our strategy for identifying Frobenius algebras will be as follows, we start by identifying an algebra A together with self-dualizing structure maps on A. We then ask "What additional condition should be satisfied for this to imply that A is a Frobenius algebra?" Well, if A *were* a Frobenius algebra, combining (4.4) and

(4.5) tells us that the coproduct could be written as both



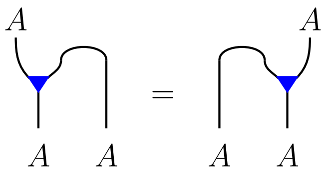
and



.

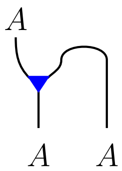
So both of these morphisms being equal is certainly a necessary condition. In fact, it is also sufficient.

PROPOSITION 4.3.9. *Let A be an algebra together with structure maps that make A self-dual. If we have*

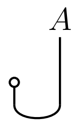


(4.6)

then equipping A with the coproduct and counit

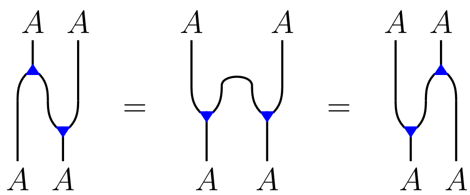


and

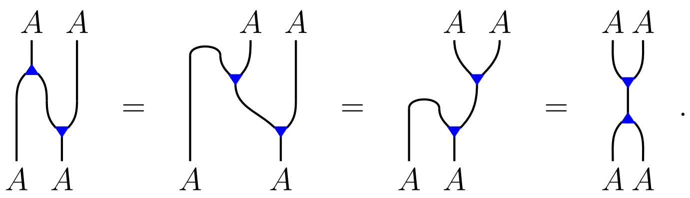


gives A the structure of a Frobenius algebra.

Proof. Checking that this does, in fact, define a coproduct, counit pair is left as an exercise for the reader. To check (4.12) we compute,



and, using the associativity of A,



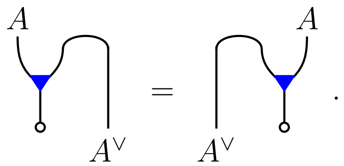
This concludes the proof.

□

DEFINITION 4.3.10. A Frobenius algebra A is called *haploid* if $\mathrm{Hom}_{\mathcal{B}}(\mathbf{1}, A) = \mathbb{K}$.

If $\mathcal{B}$ is pivotal then we also make the following definition.

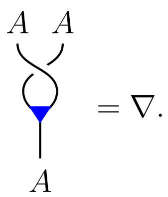
DEFINITION 4.3.11. A Frobenius algebra A is called *symmetric* if



(4.7)

If $\mathcal{B}$ is braided then we also make the following definition.

DEFINITION 4.3.12. A Frobenius algebra A is called *commutative* if its underlying algebra structure is commutative, i.e.



(4.8)

4.3.2 | DECOMPOSING THE CONDITIONS TO BE FROBENIUS

Let $\mathcal{C}$ be PTC. As our overarching goal is to prove that $\mathcal{TM}$ is a Frobenius algebra we are going to have to work with the monoidal product, braiding and pivotal structure that $\mathcal{RTC}$ inherits from $Z(\mathcal{C})$. In general this is not easy, for instance it is hard to express the tensor product of two generic objects in $\mathcal{RTC}$. However, as explained in Section 3.3.3, if we restrict our attention to functors coming from the idempotents described in Section 3.1.2 these structures may be described graphically.

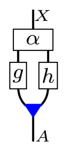
As $\mathcal{TM}$ is not of the form $(XY, \epsilon_X^Y)^\sharp$ equipping it with the structure of a Frobenius algebra directly is difficult. However, if we assume that $\mathcal{C}$ is modular then we can decompose $\mathcal{TM}$ as follows:

$$\mathcal{TM} = \bigoplus_{I,J} \mathcal{TM}_I^J \cdot \epsilon_I^J.$$

We may then define the Frobenius structure in terms of this decomposition.

To illustrate this approach let $\mathcal{B}$ be any monoidal category with complete set of simples $\mathrm{Irr}(\mathcal{B})$ and let A be an object in $\mathcal{B}$. Any morphism ∇ from $A \otimes A$ to A gives rise to the following morphisms,

$$\nabla_X^{Y,Z} \colon\ \mathrm{Hom}_{\mathcal{B}}(X, YZ) \to \mathrm{Hom}(A_Y \otimes A_Z, A_X)$$

$$\alpha \mapsto \left(g \otimes h \mapsto \right.$$  $$\left. \right)$$

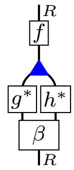
where X, Y, Z are in $\mathcal{B}$ and $A_X := \mathrm{Hom}_{\mathcal{B}}(X, A)$.

REMARK 4.3.13. The full map ∇ is determined by $\nabla_R^{S,T}$ for $R, S, T \in \mathrm{Irr}(\mathcal{C})$. Indeed we can recover it via

$$\bigoplus_{RST} \sum_{g,h,\alpha} \nabla_R^{S,T}(\alpha)(g \otimes h) \circ \alpha^* \circ (g^* \otimes h^*) = \nabla$$

where g ranges over a basis of A_S, h ranges over a basis of A_T and α ranges over a basis of $\mathrm{Hom}_{\mathcal{B}}(R, ST)$. Similarly any morphism from A to $A \otimes A$ can also be decomposed in the following way

$$\Delta_{ST}^{R}\colon\ \mathrm{Hom}_{\mathcal{B}}(ST,R)\to \mathrm{Hom}(A_R, A_S\otimes A_T)$$

$$\beta \mapsto \left(f\mapsto \sum_{g,h}$$  $$g\otimes h \right)$$

and then recovered from

$$\sum_{\substack{RST\\ \beta,f}} \Delta_{ST}^{R}(\beta)(f)\circ\beta^{*}\circ f^{*}=\Delta.$$

The goal for the remainder of this section will be to rewrite the equations that appear in Section 4.3.1 in terms of $\nabla_R^{S,T}$ and Δ_{ST}^R.

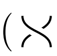
LEMMA 4.3.14. *Let A be an object in $\mathcal{B}$ and let ∇ be in $\mathrm{Hom}_{\mathcal{B}}(A\otimes A,A)$. Then ∇ is associative if*

$$\nabla_{RST}^{RS,T}(\mathrm{id})\big(\nabla_{RS}^{R,S}(\mathrm{id})(f\otimes g)\otimes h\big)=\nabla_{RST}^{R,ST}(\mathrm{id})\big(f\otimes\nabla_{ST}^{S,T}(\mathrm{id})(g\otimes h)\big) \tag{4.9}$$

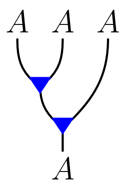
for all $R,S,T\in\mathrm{Irr}(\mathcal{B})$, $\alpha\in\mathrm{Hom}_{\mathcal{B}}(R,ST)$, $f\in A_R$, $g\in A_S$ and $h\in A_T$. An element $u\in A_{\mathbf{1}}$ is a unit for ∇ if

$$\nabla_S^{\mathbf{1},S}(\mathrm{id})(u\otimes g)=g \quad \textit{and} \quad \nabla_S^{S,\mathbf{1}}(\mathrm{id})(g\otimes u)=g \tag{4.10}$$

Furthermore, if $\mathcal{B}$ is braided then ∇ is commutative if

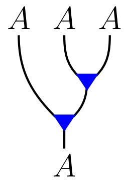
$$\nabla_{ST}^{T,S}\Big($$  $$\Big)(h\otimes g)=\nabla_{ST}^{S,T}(\mathrm{id})(g\otimes h). \tag{4.11}$$

Proof. The first claim follows from the fact that, by decomposing the top of each strand as in Lemma 2.3.3, we have



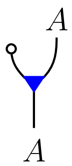
$$=\sum_{\substack{R,S,T\\ f,g,h}}\nabla_{RST}^{RS,T}(\mathrm{id})\big(\nabla_{RS}^{R,S}(\mathrm{id})(f\otimes g)\otimes h\big)\circ(f^{*}\otimes g^{*}\otimes h^{*})$$

and



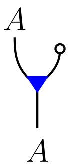
$$= \sum_{\substack{R,S,T\\ f,g,h}} \nabla_{RST}^{R,ST}(\mathrm{id})\big(f \otimes \nabla_{ST}^{S,T}(\mathrm{id})(g \otimes h)\big) \circ (f^* \otimes g^* \otimes h^*).$$

Similarly the second claim follows from



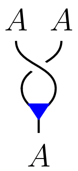
$$= \sum_{S,g} \nabla_S^{\mathbf{1},S}(\mathrm{id})(u \otimes g) \circ g^*$$

and

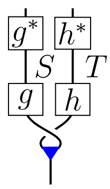


$$= \sum_{S,g} \nabla_S^{S,\mathbf{1}}(\mathrm{id})(g \otimes u) \circ g^*.$$

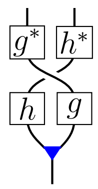
and the third claim from



$$= \sum_{\substack{S,T\\ g,h}}$$



$$= \sum_{\substack{S,T\\ g,h}}$$



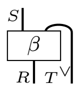
$$= \sum_{\substack{S,T\\ g,h}} \nabla_{ST}^{T,S}\Big($$

$$\Big)(h \otimes g) \circ (g^* \otimes h^*).$$

□

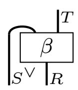
LEMMA 4.3.15. *We now suppose that $\mathcal{B}$ is a pivotal category. Let A be an algebra object in $\mathcal{B}$ (with product ∇) together with structure maps that make A self-dual. Then A satisfies* (4.6) *if and only if*

$$d(S)\left(g^* \circ \nabla_S^{R,T^\vee}\left(\right.\right.$$



$$\left.\left.\right)\right)(f \otimes (h^*)^\vee)$$

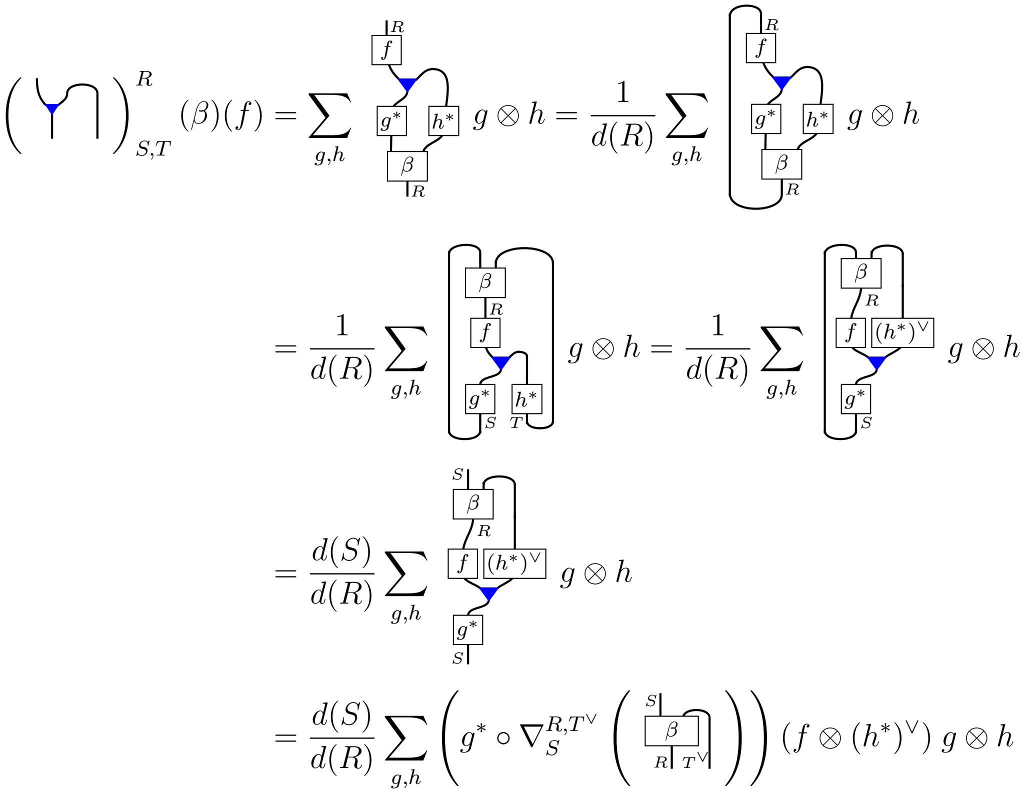
$$= d(T)\left(h^* \circ \nabla_T^{S^\vee,R}\left(\right.\right.$$



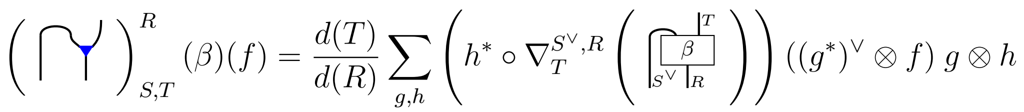
$$\left.\left.\right)\right)((g^*)^\vee \otimes f) \tag{4.12}$$

for all $R, S, T \in \mathrm{Irr}(\mathcal{B})$, $\beta \in \mathrm{Hom}_{\mathcal{B}}(ST, R)$, $h, f \in A_R$, $g, h \in A_T$ and $f, g \in A_S$.

Proof. Decomposing the coproduct given by the left-hand side of (4.6) gives



In an analogous way, we also have



which proves the proposition.

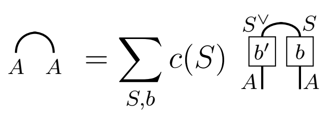
∎

Now suppose that we have a perfect pairing

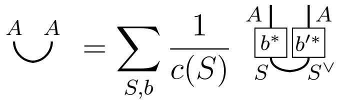
$$\langle -,- \rangle_S \colon A_S \otimes A_{S^\vee} \to \mathbb{K}$$

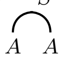
for all $S \in \mathrm{Irr}(\mathcal{B})$. We then have the following:

LEMMA 4.3.16. *Let c be a map from $\mathrm{Irr}(\mathcal{B})$ to $\mathbb{K} \setminus \{0\}$. We consider the morphisms*



and

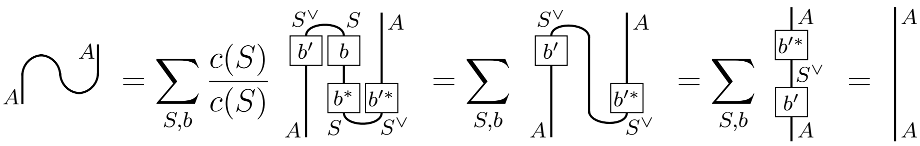


where $\{b\}$ *is a basis of* A_S *and* $\{b'\}$ *is the corresponding dual basis of* $A_{S^\vee}$ *with respect to* $\langle -,- \rangle_S$. *Then* (A,  , ) *is a dual object to* A. *Furthermore, with respect to this duality, we have*

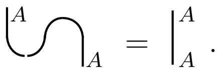
$$\langle f, (g^*)^\vee \rangle = c(S) g^*(f) \tag{4.13}$$

for all $f, g \in A_S$.

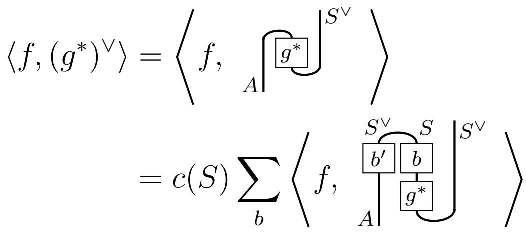
Proof. We have



and, in the same way, we also have



To prove the second claim we simply compute



$$= c(S) \sum_b g^*(b) \langle f, b' \rangle = c(S) g^*(f).$$

∎

4.3.3 | $\mathcal{TM}$ is a Frobenius Algebra

As before let $\mathcal{C}$ be an MTC and let $\mathcal{M}$ be a pivotal monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ where $\mathcal{D}$ is a pivotal monoidal category. Our first step is to equip $\mathcal{TM}$ with the structure of an algebra. We do this by specifying a map

$$\nabla_{\mathbf{X}}^{\mathbf{Y},\mathbf{Z}}\colon \mathrm{Hom}_{\mathcal{RTC}}(\mathbf{X},\mathbf{YZ}) \to \mathrm{Hom}(\mathcal{TM}_{\mathbf{Y}} \otimes \mathcal{TM}_{\mathbf{Z}}, \mathcal{TM}_{\mathbf{X}})$$

for all $\mathbf{X},\mathbf{Y},\mathbf{Z}$ in $\mathcal{RTC}$ of the form $(AB, \epsilon_A^B)^\sharp$. As $\mathcal{C}$ is modular $\{(IJ, \epsilon_I^J)^\sharp\}_{I,J\in\mathrm{Irr}(\mathcal{C})}$ forms a complete set of simples and this determines a map $\nabla\colon \mathcal{TM}\otimes\mathcal{TM} \to \mathcal{TM}$ as described in Remark 4.3.13. We recall from Proposition 4.1.1 that $\mathcal{TM}_A^B$ is identified with the subspace of $\mathrm{Hom}_D(\mathbf{1}, \mathcal{M}(AB))$ characterised by (4.1).

Definition 4.3.17. Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be given by $(AB, \epsilon_A^B)^\sharp$, $(CD, \epsilon_C^D)^\sharp$ and $(EF, \epsilon_E^F)^\sharp$ respectively. Let α be in $\mathrm{Hom}_{\mathcal{RTC}}(\mathbf{X},\mathbf{YZ}) = \mathrm{Hom}_{\mathcal{TC}}(\epsilon_A^B, \epsilon_C^D \otimes_{\mathcal{TC}} \epsilon_E^F)$. We consider the map

$$\begin{aligned}\mathrm{Hom}_D(\mathbf{1}, \mathcal{M}(CD)) \otimes \mathrm{Hom}_D(\mathbf{1}, \mathcal{M}(EF)) &\to \mathrm{Hom}_D(\mathbf{1}, \mathcal{M}(AB))\\ f\otimes g &\mapsto \mathcal{TM}(\alpha)(f \otimes_{\mathcal{D}} g).\end{aligned}$$

We note that the image of this map is in $\mathcal{TM}_{\mathbf{X}} = \mathcal{TM}_A^B$ as

$$\mathcal{TM}(\epsilon_A^B)\circ \mathcal{TM}(\alpha)(f\otimes_{\mathcal{D}} g) = \mathcal{TM}(\alpha\circ\epsilon_A^B)(f\otimes_{\mathcal{D}} g) = \mathcal{TM}(\alpha)(f\otimes_{\mathcal{D}} g).$$

Therefore restricting this map to the subspace $\mathcal{TM}_{\mathbf{Y}}\otimes\mathcal{TM}_{\mathbf{Z}}$ gives a map

$$\nabla_{\mathbf{X}}^{\mathbf{Y},\mathbf{Z}}(\alpha)\colon \mathcal{TM}_{\mathbf{Y}}\otimes\mathcal{TM}_{\mathbf{Z}} \to \mathcal{TM}_{\mathbf{X}}.$$

Let $\nabla : \mathcal{TM}\otimes\mathcal{TM}\to\mathcal{TM}$ be the map construed from $\nabla_{\mathbf{X}}^{\mathbf{Y},\mathbf{Z}}(\alpha)$ as described in Section 4.3.2.

Proposition 4.3.18. *The morphisms ∇ and*

$$u := \mathrm{id}_{\mathbf{1}_{\mathcal{D}}} \in \mathcal{TM}_{\mathbf{1}}^{\mathbf{1}} = \mathrm{Hom}_{\mathcal{RTC}}(\mathbf{1}_{\mathcal{TC}}, \mathcal{TM})$$

form a product/unit pair that make $\mathcal{TM}$ a commutative *algebra.*

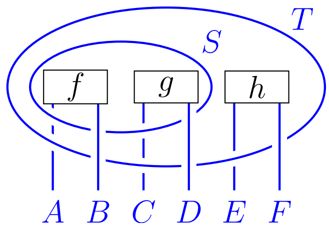
Proof. Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be given by $(AB, \epsilon_A^B)^\sharp$, $(CD, \epsilon_C^D)^\sharp$ and $(EF, \epsilon_E^F)^\sharp$ respectively.

To prove the desired result we have to show that (4.9),(4.10) and (4.11) are satisfied. We first note that (4.10) reduces to a triviality in this case.

To verify (4.9) we let f, g and h be in $\mathcal{TM}^B_A, \mathcal{TM}^D_C$ and $\mathcal{TM}^F_E$ respectively and compute,

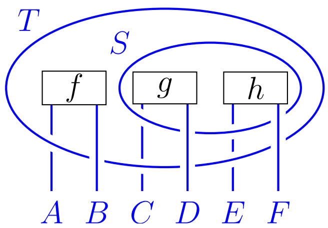
$$\nabla^{\mathbf{XY},\mathbf{Z}}_{\mathbf{XYZ}}(\epsilon^B_A \otimes_{\mathcal{TC}} \epsilon^D_C \otimes_{\mathcal{TC}} \epsilon^F_E)\big(\nabla^{\mathbf{X},\mathbf{Y}}_{\mathbf{XY}}(\epsilon^B_A \otimes_{\mathcal{TC}} \epsilon^D_C)(f \otimes g) \otimes h\big)$$

$$= \frac{1}{d(\mathcal{C})^2}\sum_{S,T} d(S)d(T)$$



$$= f \otimes_D g \otimes_D h$$

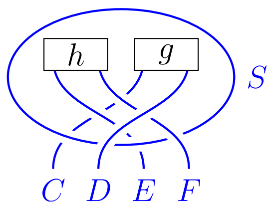
$$= \frac{1}{d(\mathcal{C})^2}\sum_{S,T} d(S)d(T)$$



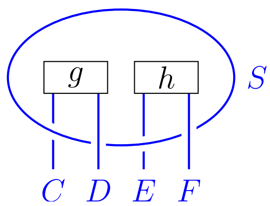
$$= \nabla^{\mathbf{X},\mathbf{YZ}}_{\mathbf{XYZ}}(\epsilon^B_A \otimes_{\mathcal{TC}} \epsilon^D_C \otimes_{\mathcal{TC}} \epsilon^F_E)\big(f \otimes \nabla^{\mathbf{Y},\mathbf{Z}}_{\mathbf{YZ}}(\epsilon^D_C \otimes_{\mathcal{TC}} \epsilon^F_E)(g \otimes h)\big)$$

where we are simply using multiple instances of Proposition 4.1.1. Finally, once again by Proposition 4.1.1, we have

$$\nabla^{\mathbf{Z},\mathbf{Y}}_{\mathbf{YZ}}(\times)(h \otimes g) = \frac{1}{d(\mathcal{C})}\sum_{S} d(S)$$



$$= \frac{1}{d(\mathcal{C})}\sum_{S} d(S)$$



$$= \nabla^{\mathbf{Y},\mathbf{Z}}_{\mathbf{YZ}}(\epsilon^D_C \otimes_{\mathcal{TC}} \epsilon^F_E)(g \otimes h)$$

which proves (4.11).

□

The next step is to equip $\mathcal{TM}$ with self-dualizing structure maps. For this to work we need to make some additional assumptions on $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$. Firstly we assume that the idempotent completion of $\mathcal{D}$, denoted $\overline{\mathcal{D}}$, is a multifusion category. Secondly we assume that $\overline{\mathcal{M}}\colon \mathcal{C} \to \overline{\mathcal{D}}$, obtained by composing $\mathcal{M}$ with this embedding, is *indecomposable*. In other words, that there do not exist functors $\overline{\mathcal{M}}_1\colon \mathcal{C} \to \overline{\mathcal{D}}_1$ and $\overline{\mathcal{M}}_2\colon \mathcal{C} \to \overline{\mathcal{D}}_2$ such that $\overline{\mathcal{M}} = \overline{\mathcal{M}}_1 \oplus \overline{\mathcal{M}}_2$, where $\overline{\mathcal{D}}_i \leq \overline{\mathcal{D}}$.

As described in Section 2.2.13, $\overline{\mathcal{D}}$ decomposes into $\bigoplus_{i,j\in I} {}_i\overline{\mathcal{D}}_j$ where I is an indexing set for the primitive idempotents in $\mathrm{End}_{\overline{\mathcal{D}}}(\mathbf{1})$. Therefore the condition that $\overline{\mathcal{M}}$ is indecomposable is equivalent to requiring that there exists no subset $K \subset I$ such that ${}_i\overline{\mathcal{M}}(X)_j = {}_j\overline{\mathcal{M}}(X)_i = 0$ for all X in $\mathcal{C}$, $i \in K$ and $j \in I \setminus K$.

PROPOSITION 4.3.19. *$\overline{\mathcal{M}}$ is indecomposable if and only if $\mathcal{TM}_{\mathbf{1}}^{\mathbf{1}} = \mathbb{K}$. Furthermore, in this case, any non-zero $\alpha \in \mathcal{TM}_X^Y \leq \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(XY))$ has a left-inverse in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(XY), \mathbf{1})$ for all X, Y in $\mathcal{C}$.*

Proof. By Proposition 4.1.1, $\mathcal{TM}_{\mathbf{1}}^{\mathbf{1}}$ is given by the subspace of $\mathrm{End}_{\mathcal{D}}(\mathbf{1})$ such that

$$\alpha \otimes \mathrm{id}_{\mathcal{M}(Z)} = \mathrm{id}_{\mathcal{M}(Z)} \otimes \alpha \quad \forall Z \text{ in } \mathcal{C}.$$

Embedding this equality into $\overline{\mathcal{D}}$ and decomposing gives

$$\alpha_i \, \mathrm{id}_{{}_i\overline{\mathcal{M}}(Z)_j} = \alpha_j \, \mathrm{id}_{{}_i\overline{\mathcal{M}}(Z)_j} \quad \forall Z \text{ in } \mathcal{C}.$$

This implies $\alpha_i = \alpha_j$ for all $i, j \in I$ if and only if $\overline{\mathcal{M}}$ is indecomposable. This proves the first claim.

To prove the second claim we recall the characterisation of $\mathcal{TM}_X^Y$ provided by Proposition 4.1.1, i.e. the subspace of $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(XY))$ such that

$$\phi \circ (\alpha \otimes \mathrm{id}_{\mathcal{M}(Z)}) = \mathrm{id}_{\mathcal{M}(Z)} \otimes \alpha \quad \forall Z \text{ in } \mathcal{C}.$$

where ϕ is a certain isomorphism. Embedding this equality into $\overline{\mathcal{D}}$ and decomposing gives

$$ {}_i\phi_j \circ (\alpha_i \otimes \mathrm{id}_{{}_i\overline{\mathcal{M}(Z)}_j}) = \mathrm{id}_{{}_i\overline{\mathcal{M}(Z)}_j} \otimes \alpha_j \quad \forall Z \text{ in } \mathcal{C}. $$

Therefore, if $\overline{\mathcal{M}}$ is indecomposable, $\alpha_i = 0$ for any $i \in I$ implies $\alpha = 0$. This proves the second claim.

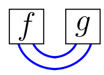
$\square$

We are now ready to equip $\mathcal{TM}$ with some self-dualizing structure maps. To accomplish this we shall use Lemma 4.3.16. We therefore first establish the following perfect pairing.

LEMMA 4.3.20. *Let X and Y be in $\mathcal{C}$. As usual $\mathcal{TM}_X^Y$ is identified with a subspace of* $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(XY))$*, however, as described in Remark 4.1.2 we identify $\mathcal{TM}_{X^\vee}^{Y^\vee}$ with a subspace of* $\mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(Y^\vee X^\vee))$*. The map*

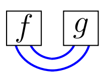
$$ \langle -, - \rangle \colon \mathcal{TM}_X^Y \otimes \mathcal{TM}_{X^\vee}^{Y^\vee} \to \mathcal{TM}_{\mathbf{1}}^{\mathbf{1}} = \mathbb{K} $$

$$ f \otimes g \mapsto $$



is a perfect pairing.

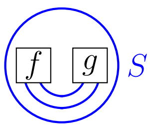
Proof. Given a non-zero $f \in \mathcal{TM}_X^Y$, by Proposition 4.3.19 there exists $g \in \mathrm{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(Y^\vee X^\vee))$ such that



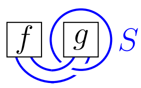
$$ = \mathrm{id}_{\mathbf{1}} . $$

We therefore have

$$ \mathrm{id}_{\mathbf{1}} = \frac{1}{d(\mathcal{C})} \sum_S d(S) $$



$$ = \frac{1}{d(\mathcal{C})} \sum_S d(S) $$



(4.14)

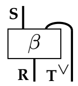
by Proposition 4.1.1. We now consider $\tilde{g} = \mathcal{TM}(\tilde{\epsilon}_{X^\vee}^{Y^\vee})(g) \in \mathcal{TM}_{X^\vee}^{Y^\vee}$ (where $\tilde{\epsilon}_{X^\vee}^{Y^\vee}$ is given by (3.5)). Then the right-hand side of (4.14) is $\langle f, \tilde{g} \rangle$ and so we are done.

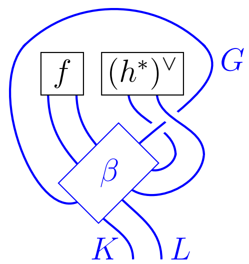
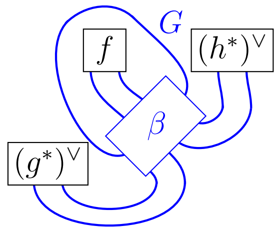
$\square$

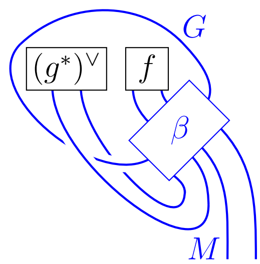
REMARK 4.3.21. We note that this perfect pairing is *symmetric* with respect to the pivotal structure, i.e. $\langle f, g\rangle = \langle g, f\rangle$ where $f \in \mathcal{TM}_X^Y = \mathcal{TM}_{X^{\vee\vee}}^{Y^{\vee\vee}}$.

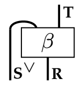
PROPOSITION 4.3.22. *We consider $\mathcal{TM}$ equipped with the algebra structure from Proposition 4.3.18. We also equip $\mathcal{TM}$ with the self-dualizing maps given by Lemma 4.3.20 and Lemma 4.3.16 with $c = d$ (the dimension map for $\mathcal{RTC}$). Then $\mathcal{TM}$ satisfies* (4.3)*, i.e. is a Frobenius algebra.*

Proof. Let $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ be given by $(IJ, \epsilon_I^J)^\sharp, (KL, \epsilon_K^L)^\sharp$ and $(MN, \epsilon_M^N)^\sharp$ respectively where $I, J, K, L, M, N \in \mathrm{Irr}(\mathcal{C})$. Let f, g and h be in $\mathcal{TM}_I^J, \mathcal{TM}_K^L$ and $\mathcal{TM}_M^N$ respectively and let β be in $\mathrm{Hom}_{\mathcal{RTC}}(\mathbf{ST}, \mathbf{R}) = \mathrm{Hom}_{\mathcal{TC}}(\epsilon_K^L \otimes_{\mathcal{TC}} \epsilon_M^N, \epsilon_I^J)$. We have

$d(\mathbf{S})\left(g^* \circ \nabla_{\mathbf{S}}^{\mathbf{R},\mathbf{T}^\vee}\left(\right.\right.$  $\left.\left.\right)\right)(f \otimes (h^*)^\vee)$

$= d(\mathbf{S})\, g^* \left(\right.$  $\left.\right) =$ 

$= d(\mathbf{T})\, h^* \left(\right.$  $\left.\right)$

$= d(\mathbf{T})\left(h^* \circ \nabla_{\mathbf{T}}^{\mathbf{S}^\vee,\mathbf{R}}\left(\right.\right.$  $\left.\left.\right)\right)((g^*)^\vee \otimes f)$

where we have used Proposition 4.1.1 and Lemma 4.3.16 multiple times.

$\square$

THEOREM 4.3.23. *Let $\mathcal{C}$ be an MTC and let $\mathcal{M}$ be a pivotal tensor functor from $\mathcal{C}$ to $\mathcal{D}$ such that $\overline{\mathcal{M}}$ is indecomposable. Then $\mathcal{TM}$ is a haploid, symmetric, commutative Frobenius algebra.*

Proof. This result follows from Proposition 4.3.19, Proposition 4.3.18, Proposition 4.3.22 and Remark 4.3.21.

□

4.4 | Connection with α-Induction

4.4.1 | Module Categories over $\mathcal{C}$ and α-Induction

DEFINITION 4.4.1. Let $\mathcal{C}$ be a monoidal category. A *module category* over $\mathcal{C}$ is a monoidal category $\mathcal{B}$ together with a monoidal functor $\mathcal{M}\colon \mathcal{C} \to \mathrm{End}(\mathcal{B})$, where $\mathrm{End}(\mathcal{B})$ is the category of endofunctors on $\mathcal{B}$. If $\mathcal{B}$ is semisimple with finitely many simple objects we call $\mathcal{M}$ a *finite* module category over $\mathcal{C}$.

Let $\mathcal{M}\colon \mathcal{C} \to \mathrm{End}(\mathcal{B})$ be a finite module category and let $\mathrm{Irr}(\mathcal{B})$ be a complete set of simples in $\mathcal{B}$. We then consider

$$T := \bigoplus_{v\in \mathrm{Irr}(\mathcal{C})} v \in \mathrm{Obj}(\mathcal{B})$$

and the semisimple algebra $A = \mathrm{End}_{\mathcal{B}}(T)$.

REMARK 4.4.2. As every object in $\mathrm{Irr}(\mathcal{C})$ is simple and distinct, Schur's Lemma implies that A is a direct sum of division algebras over $\mathbb{K}$.

As T is a projective generator in $\mathcal{B}$ the (covariant) functor

$$\mathrm{Hom}_{\mathcal{B}}(T, -)\colon \mathcal{B} \to \text{Mod-}A$$

is an equivalence of categories. A finite module category over $\mathcal{C}$ is therefore equivalent to a monoidal functor

$$\mathcal{M}\colon \mathcal{C} \to \mathrm{End}(\text{Mod-}A) = A, A\text{-Bimod}\,.$$

From a physical point of view Cardy [Car89] showed that the algebraic data of an *annular partition function* in a boundary (rational) conformal field theory is given by a finite module category over the corresponding MTC. The process know as α-induction is an operator algebra technique developed by Böckenhauer and Evans [BE98] that produces a *toroidal partition function* (as described in the introduction) from an annular partition function. Ostrik [Ost03, Section 5] rephrased α-induction using categorical language in the following way.

Let $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$ be a finite module category over a PTC $\mathcal{C}$, where $\mathcal{D}$ denotes $A, A\text{-}\mathrm{Bimod}$. For A, B in $\mathcal{C}$ we consider the subspace

$$\mathrm{Hom}^{\sigma}_{\mathcal{M}}(A,B) \leq \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(A), \mathcal{M}(B))$$

defined by the condition that $\beta \in \mathrm{Hom}_{\mathcal{D}}(M(A), M(B))$ satisfies, for all X in $\mathcal{C}$,

$$\begin{tikzcd}
\mathcal{M}(A)\otimes\mathcal{M}(X) \arrow[r, "\mathcal{M}(\sigma_{AY})"] \arrow[d, "\beta\otimes\mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}(Y)\otimes\mathcal{M}(A) \arrow[d, "\mathrm{id}\otimes\beta"] \\
\mathcal{M}(B)\otimes\mathcal{M}(X) \arrow[r, "\mathcal{M}(\overline{\sigma}_{BY})"] & \mathcal{M}(X)\otimes\mathcal{M}(B)
\end{tikzcd} \tag{4.15}$$

where σ and $\overline{\sigma}$ are the braiding on $\mathcal{C}$ and its opposite respectively. The principal claim of α-induction is then as follows. Under the assumption that the dimensions of all the objects in $\mathcal{C}$ are positive, the $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$-matrix whose entries are given by the dimension of $\mathrm{Hom}^{\sigma}_{\mathcal{M}}(I^{\vee}, J)$ commutes with the modular data of $\mathcal{C}$. Furthermore if $\mathcal{M}$ is irreducible then this matrix is a *modular invariant* (see Definition 1.1.1).

REMARK 4.4.3. The claim found in [Ost03] is actually that the $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$-matrix whose entries are given by the dimension of $\mathrm{Hom}^{\sigma}_{\mathcal{M}}(I, J)$ commutes with the modular data of $\mathcal{C}$. However as the modular data always commutes with the charge conjugation matrix (see Remark 2.2.26) these statements are equivalent.

REMARK 4.4.4. As the proof of this claim is given in the operator algebra context it should only be considered a theorem in the case when the categories and module categories under consideration may be realised within that context.

In [Ost03] Ostrik also provides the following example to prove the necessity of the condition that the objects in $\mathcal{C}$ have positive dimension.

EXAMPLE 4.4.5. Let $\mathcal{C}$ be the fusion category with complete set of simples $\{\underline{0}, \underline{1}\}$, where $\underline{0}$ is the tensor unit and $\underline{1} \otimes \underline{1} = \underline{0}$. As this category is rigid we may equip it with the pivotal structure $\delta_{\underline{1}} = -\operatorname{id}_{\underline{1}}$ (so that $d(\underline{1}) = -1$). One may also check that setting $\sigma_{\underline{11}} = \operatorname{id}_{\underline{2}}$ defines a (degenerate) braiding on the category and we obtain a PTC. We then consider the module category

$$\begin{aligned} \mathcal{M} \colon \mathcal{C} &\to \underline{\mathrm{Vect}} \\ \underline{0} &\mapsto \mathbb{K} \\ \underline{1} &\mapsto \mathbb{K}. \end{aligned}$$

As the braiding is given by the identity, we have $\sigma = \overline{\sigma}$ and Equation (4.15) reduces to a tautology. Therefore $\operatorname{Hom}^{\sigma}_{\mathcal{M}}(\underline{0}, \underline{1}) = \mathbb{K}$ and the resulting dimension matrix fails to commute with the T-matrix

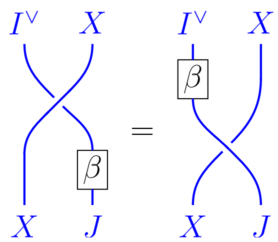
$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4.4.2 | CONNECTION WITH $\mathcal{TM}$

We start by remarking that Condition 4.15 makes sense even when $\mathcal{D}$ is an arbitrary tensor category. Therefore to connect Ostrik's formulation of α-induction to $\mathcal{TM}$ we have the following.

THEOREM 4.4.6. *Let $\mathcal{C}$ be a PTC and let $\mathcal{M} \colon \mathcal{C} \to \mathcal{D}$ be a pivotal monoidal functor. Then* $\operatorname{Hom}^{\sigma}_{\mathcal{M}}(I^{\vee}, J) \cong \mathcal{TM}^{J}_{I}$.

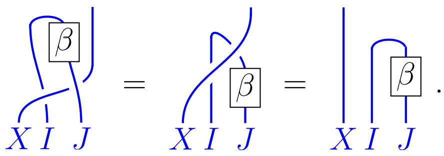
Proof. Graphically Condition (4.15) is given by



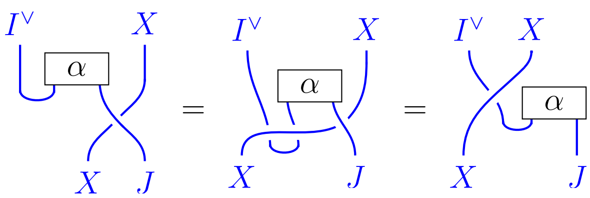
(4.16)

for all X in $\mathcal{C}$. As $\mathcal{TM}^{J}_{I}$ is a subspace of $\operatorname{Hom}_{\mathcal{D}}(\mathbf{1}, \mathcal{M}(IJ)) = \operatorname{Hom}_{\mathcal{D}}(\mathcal{M}(I^{\vee}), \mathcal{M}(J))$ we only have to check that Condition (4.16) is equivalent to Condition (4.1).

Suppose $\beta \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(I), \mathcal{M}(J))$ satisfies Condition (4.15). Then we have



Furthermore, for $\alpha \in \mathcal{TM}_I^J$, we have



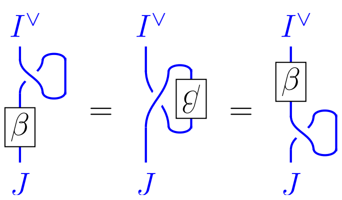
where the final equality uses Proposition 4.1.1. This is equivalent to Condition (4.15) as desired.

$\square$

The alternative characterisation of $\mathcal{TM}_I^J$ given by Theorem 4.4.6 allows for the following generalization of Corollary 4.2.5 to the pre-modular case.

THEOREM 4.4.7. *Let $\mathcal{C}$ be a PTC, let $\mathcal{D}$ be a pivotal monoidal category and let $\mathcal{M} \colon \mathcal{C} \to \mathcal{D}$ be a pivotal monoidal functor. Then $\mathcal{TM}$ is T-invariant.*

Proof. Let I, J be such that $\mathcal{TM}_I^J \neq 0$. Then, by Theorem 4.4.6, there exists a non-zero map $\beta \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(I^\vee), \mathcal{M}(J))$ that satisfies (4.16). We have

$\mathcal{T}_{II}\beta =$  $= \mathcal{T}_{JJ}\beta$

where $\mathcal{T}$ denotes the T-matrix and to make certain string manipulations clearer, we have chosen to write β upside-down instead of writing $\beta^\vee$. Therefore $Z(\mathcal{TM})_{IJ} \neq 0$ implies $\mathcal{T}_{II} = \mathcal{T}_{JJ}$. As $\mathcal{T}$ is diagonal that is precisely the condition that $Z(F)$ commutes with $\mathcal{T}$.

$\square$

Let us suppose that $\mathbb{K}$ is algebraically closed and that $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$ is a module category where $\mathcal{D} = A, A\text{-Bimod}$. By Remark 4.4.2, A will be a direct sum of copies of $\mathbb{K}$, we write this decomposition as $A = \bigoplus_{i \in A_0} \mathbb{K}$. We may now describe $\mathcal{D}$ as the category whose objects are $A_0 \times A_0$-matrices of vector spaces and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{D}}(B, C) = \bigoplus_{i,j \in A_0} \mathrm{Hom}({}_iB_j, {}_iC_j).$$

For B and C in $\mathcal{D}$ the tensor product $B \otimes C$ is given by

$${}_i(B \otimes C)_j = \bigoplus_{k \in A_0} {}_iB_k \otimes {}_kC_j.$$

This category does come equipped with a natural pivotal structure. Indeed, for B in $\mathcal{D}$ we have the bimodule B^* which is given by ${}_i(B^*)_j = {}_jB_i{}^*$. As B^* is both a left and a right dual this construction gives us a pivotal structure. In general, however, $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$ will not be pivotal with respect to this standard pivotal structure. Indeed, if $\mathcal{M}$ were pivotal we would have $d(X) = d(\mathcal{M}(X))$. However, the dimension of an object B in $A, A\text{-Bimod}$ is

$$\begin{aligned} d(B)\colon A &\to A \\ a_i &\mapsto \bigoplus_j \dim({}_iB_j)a_j. \end{aligned}$$

This is only a multiple of the identity when B is a direct sum of copies of A and even then it has to be an integer multiple of the identity. Therefore, if we are to study interesting module categories we cannot require that they be pivotal. However, we can study interesting examples if we only require that $\mathcal{M}$ *induce a pivotal structure on its full image*.

Let $\mathcal{D}$ now be the full image of $\mathcal{M}$ in $A, A\text{-Bimod}$. Clearly $\mathcal{D}$ is a rigid monoidal category. Furthermore, it comes with a natural candidate pivotal structure: $\mathcal{M}(\delta_X)$, where $\delta\colon {}^\vee- \to -^\vee$ gives the pivotal structure on $\mathcal{C}$ (see Section 2.2.2). As $\mathcal{M}$ is a functor, $\mathcal{M}(\delta_X)$ is natural with respect to morphisms in $\mathcal{C}$; however, to give a pivotal structure on $\mathcal{D}$ it must be natural with respect *all* morphisms in $\mathcal{D}$. In other words the diagram

$$\begin{tikzcd}
\mathcal{M}(Y^\vee) \arrow[r, "\alpha^\vee"] \arrow[d, "\mathcal{M}(\delta_Y)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}(X^\vee) \arrow[d, "\mathcal{M}(\delta_X)"] \\
\mathcal{M}({}^\vee Y) \arrow[r, "{}^\vee\alpha"] & \mathcal{M}({}^\vee X)
\end{tikzcd} \tag{4.17}$$

must commute for all $\alpha \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(X), \mathcal{M}(Y))$. When this is satisfied and $\mathcal{D}$ is equipped with the resulting pivotal structure, the functor $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$ is automatically pivotal. We may therefore construct $\mathcal{TM}$ and Theorem 4.4.6 guarantees that $Z(\mathcal{TM})$ will give the same matrix as α-induction. Furthermore, the inclusion $\mathcal{D} \hookrightarrow A, A\text{-Bimod}$ fully embeds $\mathcal{D}$ into a multifusion category such that the notion of indecomposability described in Section 4.3.3 coincides with the standard notion of indecomposability for module categories. We therefore obtain the following corollary of Theorem 4.3.23.

COROLLARY 4.4.8. *Let $\mathcal{C}$ be an MTC over an algebraically closed field and let $\mathcal{M}\colon \mathcal{C} \to S, S\text{-Bimod}$ be an indecomposable module category over $\mathcal{C}$ that induces a pivotal structure on its full image. Then $\mathcal{TM}$ is a haploid, symmetric, commutative Frobenius algebra.*

REMARK 4.4.9. Ostrik's Example 4.4.5 also shows that the condition that $\mathcal{M}$ be pivotal is necessary for the results of Section 4.2 and Section 4.3. Indeed, one may check that his example fails to induce a pivotal structure on its full image.

For the remainder of this thesis we shall develop certain interesting examples of module categories that induce a pivotal structure on their full images.

CHAPTER 5

CASE STUDY: THE TEMPERLEY-LIEB CATEGORY

The principal purpose of this chapter is to provide an interesting class of examples upon which to apply the theory developed throughout the previous chapters. Module categories are a good source of examples of monoidal functors; however, to apply the $\mathcal{TM}$ construction we must require that they induce a pivotal structure on their full image. In this chapter we study a class of module categories that satisfy this condition.

Section 5.1 serves as an introduction to the Temperley-Lieb category, denoted TL, a $\mathbb{C}$-linear category that depends on a parameter $\beta \in \mathbb{C} \setminus \{0\}$. This is a diagrammatic category with a basis of the Hom-spaces being given by the different ways of pairing dots in such a way that a planarity condition is satisfied. For certain values of the parameter β, TL admits a proper tensor ideal. Quotienting by this ideal and taking the idempotent completion gives a fusion category, denoted $\mathcal{C}$, that may be equipped with the structure of a pre-modular tensor category. Finally it is explained how, under an additional restriction on the parameter β, $\mathcal{C}$ admits a modular structure that is equivalent to the modular structure of the category of integrable, highest weight modules of $A_1^{(1)}$ at a certain level.

Section 5.2 starts by giving a recipe to produce module categories over $\mathcal{C}$, the input of which is a symmetric quiver satisfying certain conditions. The resulting module categories possess the important property of inducing a pivotal structure on their full image. A classification result of Entingof and Ostrik [EO04, Theorem 3.12] then implies that all module categories over $\mathcal{C}$ may be thought of as resulting from this recipe and the indecomposable ones may be obtained via the double Dynkin quivers of type A,D and E. Finally, the $\mathcal{TM}$ framework is used to provide a new explanation of the appearance of an A-D-E pattern in the Cappelli-Itzykson-Zuber classification of $A_1^{(1)}$ modular invariants through a connection with the above mentioned classification of indecomposable module categories over $\mathcal{C}$.

5.1 | DEFINITION OF THE TEMPERLEY-LIEB CATEGORY

5.1.1 | BRAUER DIAGRAMS

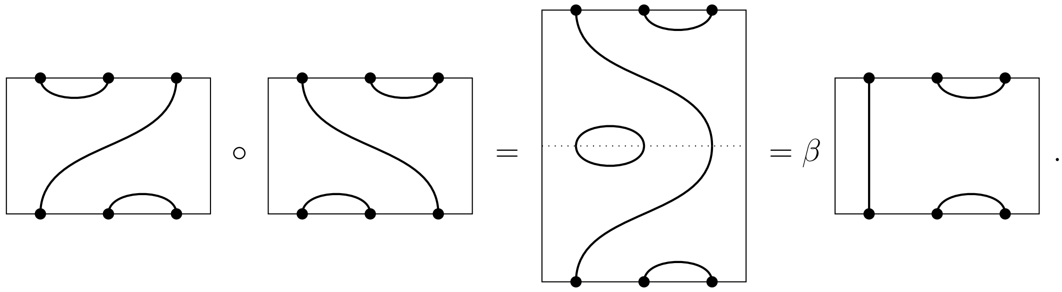
Let m and n be non-negative integers of the same parity. A *planar Brauer diagram* from m to n consists of the following:

1. A closed rectangle R in the plane with two opposite edges designated as top and bottom,
2. m marked points on the top edge and n marked points on the bottom edge,
3. $\frac{m+n}{2}$ smooth curves in R such that the curves are pairwise non-intersecting and such that for each curve γ, the set $\gamma \cap \partial R$ consists of two of the $n+m$ marked points.

Two such diagrams are equivalent if they induce the same pairing on the $n+m$ marked points.

DEFINITION 5.1.1. Let β be in $\mathbb{C} \setminus \{0\}$. The *Temperley-Lieb category* TL(β) is a $\mathbb{C}$-linear category whose objects are columns of n dots (for n natural number) denoted $\underline{n}$. A basis for $\mathrm{Hom}_{\mathrm{TL}}(\underline{m}, \underline{n})$ is the set of equivalence classes of planar Brauer diagrams from $\underline{m}$ to $\underline{n}$.

Let A be a planar Brauer diagram from m to n and let B be a planar diagram from n to k. The composition $B \circ A$ is found by identifying the bottom edge of A with the top edge of B and replacing every closed loop thus formed by a factor of β. For example,



The Temperley-Lieb category is a monoidal category with tensor product on objects defined by $\underline{m} \otimes \underline{n} = \underline{m+n}$. The tensor product of two planar Brauer diagrams is the planar Brauer diagram obtained by juxtaposing the two diagrams horizontally.

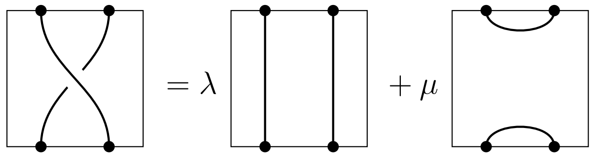
5.1.2 | Temperley-Lieb as a Braided Category

Before discussing a braiding on TL we give a brief introduction to the *quantum integers*. For $q \in \mathbb{C}^*$ the quantum integers are defined inductively via

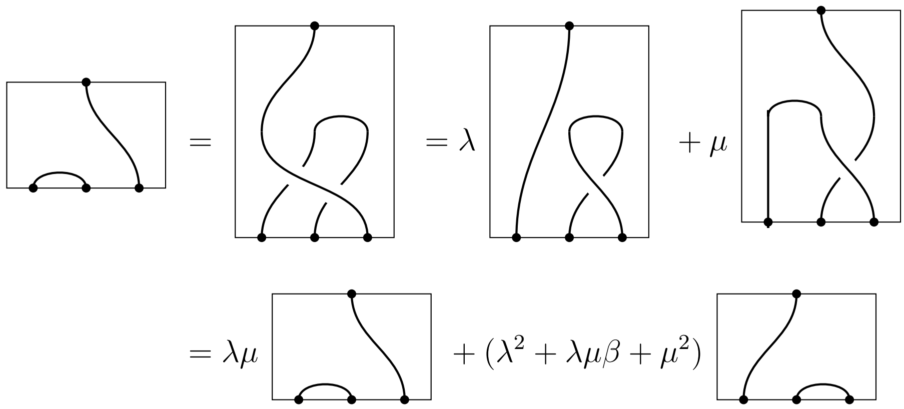
$$[1]_q = 1,\ [2]_q = q + q^{-1}$$
$$[k+1]_q = [2]_q[k]_q - [k-1]_q.$$

We note that for $q = 0$ the quantum integers coincide with the classical integers and that $[k]_q = 0$ if and only if $q^{2k} = 1$.

The goal of this Section is to introduce a braiding on TL. We do this by first considering the braiding restricted to single strands. As $\mathrm{End}(\underline{2})$ is 2-dimensional we must have



for some λ, μ in $\mathbb{C}$. The braiding axioms then imply



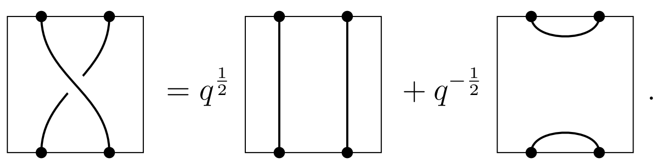
giving us the conditions

$$0 \neq \mu = \lambda^{-1} \quad \text{and} \quad \beta = -\lambda^2 - \lambda^{-2}.$$

Therefore for TL to admit a braiding it must have parameter

$$\beta = -q - q^{-1} = -[2]_q \tag{5.1}$$

for some $q \in \mathbb{C}^*$. In this case the braiding on single strands is given by

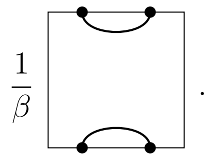


(5.2)

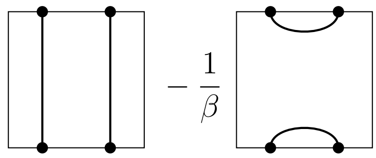
As every object in TL is a tensor power of $\underline{1}$ this entirely determines the braiding. That (5.2) does indeed define a braiding on TL under no additional conditions on β is proved in Chapter XII of [Tur16].

5.1.3 | Temperley-Lieb and Semisimplicity

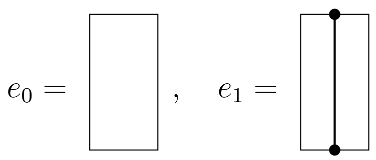
We start by observing that TL is a good example of a category that fails to be idempotent complete. We consider, for example, the following idempotent on $\underline{2}$,

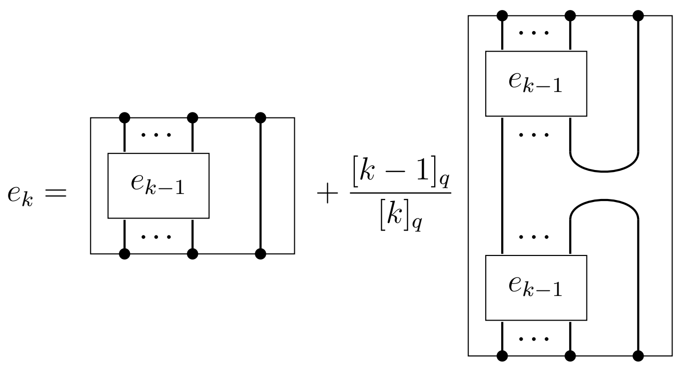


This idempotent clearly has image object $\underline{0}$. However the complementary idempotent



does not admit an image object as there is no X in TL such that $\underline{2} = \underline{0} \oplus X$. The relevant question is therefore whether or not TL admits a complete set of primitive orthogonal idempotents. For $k \in \mathbb{N}$ we consider the idempotent $e_k \in \mathrm{End}(\underline{k})$ defined inductively by





where q is the element of $\mathbb{C}^*$ such that $\beta = -[2]_q$ (see Section 5.1.2). These are called the *Jones-Wenzl idempotents* and were first introduced in [Wen87]. We note that e_k is not well defined if

$$[k]_q = 0 \iff q^{2k} = 1. \tag{5.3}$$

Let us first suppose that q is *not* a root of unity such that (5.3) never occurs (this is called the *generic* case). Let X_k denote the object $(\underline{k}, e_k)^\sharp$ in the category of representations of TL. The following theorem is proved in [Coo07].

THEOREM 5.1.2. *The X_k are Schurian and form a complete set of simples in $\mathcal{R}$TL. Furthermore they satisfy the fusion rules*

$$\begin{aligned} X_0 \otimes X_1 &= X_1, \\ X_k \otimes X_1 &= X_{k-1} \oplus X_{k+1}. \end{aligned}$$

A category is said to be of *semisimple type* if its category of representations is semisimple. Therefore in the generic case TL is of semisimple type. From now on we shall suppose that q is a root of unity (this is called the singular case). Let h be the smallest positive integer such that $q^{2h} = 1$ or equivalently $[h]_q = 0$. We call h the *Coxeter number* of q.

In the singular case TL is *not* of semisimple type. However in [GW02] it is shown that TL admits precisely one proper tensor ideal (a property exclusive to the singular case). They also showed that the ideal is generated by e_{h-1}, the last Jones-Wenzl idempotent it is possible to define.

DEFINITION 5.1.3. The *reduced Temperley-Lieb category* $\mathrm{TL}^{\mathrm{red}}$ is defined as the quotient of TL by the unique proper tensor ideal generated by e_{h-1}.

Let π denote the quotient functor $\pi\colon \mathrm{TL} \to \mathrm{TL}^{\mathrm{red}}$. For $0 \leq k \leq h-2$, let X_k now denote the image object associated to the idempotent $(\pi(\underline{k}), \pi(e_k))^\sharp$ in the idempotent completion of $\mathrm{TL}^{\mathrm{red}}$. The following theorem is also proved in [Coo07].

THEOREM 5.1.4. *The X_k are Schurian and form a complete set of simples in $\mathcal{R}\mathrm{TL}^{red}$.*

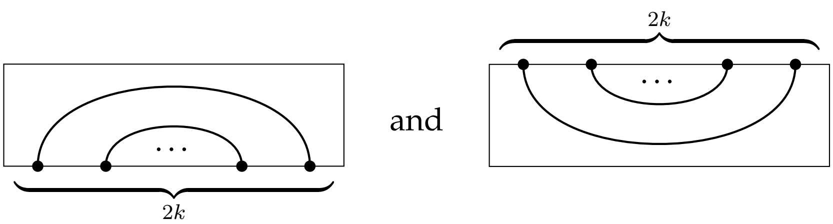
Furthermore they satisfy the fusion rules

$$X_0 \otimes X_1 = X_1$$
$$X_{h-2} \otimes X_1 = X_{h-3}$$
$$X_k \otimes X_1 = X_{k-1} \oplus X_{k+1}, \qquad 1 \leq k \leq h-3.$$

5.1.4 | TEMPERLEY-LIEB AS A MODULAR TENSOR CATEGORY

As $\mathcal{R}\mathrm{TL}^{\mathrm{red}}$ will be the category under consideration for the remainder of this Section we shall simply denote it by $\mathcal{C}$.

We have already seen that $\mathcal{C}$ is a semisimple monoidal category with finitely many isomorphism classes of simple objects. TL is rigid as $\underline{k}$ is self dual with creation and annihilation maps



respectively. This notion of duality is preserved by the quotient described in Definition 5.1.3 and extends naturally to $\mathcal{C}$. Therefore $\mathcal{C}$ is rigid and every object in $\mathcal{C}$ is self dual. Self duality also gives $\mathcal{C}$ the natural pivotal structure

$$\delta_X = \mathrm{id}_X \quad \forall X \text{ in } \mathcal{C}. \tag{5.4}$$

In Section 5.1.2 we discussed how to equip TL with a braiding, which $\mathcal{C}$ also inherits. Chapter XII of [Tur16] also proves that this braiding is balanced with respect to (5.4). Therefore the only condition that can prevent $\mathcal{C}$ from being an MTC is the non-degeneracy of the braiding. We have the following result,

THEOREM 5.1.5 ([Tur16], Theorem 7.5.3.). *Suppose there exists $r \geq 3$ such that the roots appearing in the braiding* (5.2) *are primitive $4r$ roots of unity (in particular this implies that q is a primitive $2r$ root of unity). Then $\mathcal{C}$ is a MTC.*

So what does the modular data look like in this case? Rather surprisingly it looks a lot like the modular data seen in the introduction (see Section 1.1). More explicitly, we have (up to a scalar)

$$\mathcal{S}_{ab} = (-1)^{a+b}\sqrt{\frac{2}{h}}\sin\left(\pi\frac{ab}{h}\right)$$

and

$$\mathcal{T}_{ab} = (-1)^{a-1}\exp\left(\pi i\frac{a^2}{2h}\right)\delta_{a,b}$$

where $a, b \in \mathcal{I} = \{1, 2, ..., h-1\}$ indexes the simples in $\mathcal{C}$ via $k \mapsto X_{k-1}$. Apart from the -1 factors we have exactly recovered the Kac-Peterson matrices for $A_1^{(1)}$. This is due to the fact that $\mathrm{Rep}_{h-2}\, A_1^{(1)}$ is equivalent to $\mathcal{C}$ as a modular tensor category apart from one small difference: they have different pivotal structures. To recover a category that is pivotally equivalent to $\mathrm{Rep}_{h-2}\, A_1^{(1)}$ one would have to induce onto $\mathcal{C}$ the pivotal structure on TL given by

$$\delta_{\underline{n}} = \begin{cases} \mathrm{id}_{\underline{n}} & \text{for } n \text{ even} \\ -\mathrm{id}_{\underline{n}} & \text{for } n \text{ odd} \end{cases} \qquad \forall n \in \mathbb{N},$$

as opposed to (5.4). Alternatively one could consider the so-called "disoriented" diagrammatic category presented in [CMW09, p. 5]. This tweaked version of $\mathcal{C}$ come naturally equipped with a pivotal structure that is compatible with $\mathrm{Rep}_{h-2}\, A_1^{(1)}$. For further details on this difference of pivotal structures see [ST09].

REMARK 5.1.6. We note that M is a modular invariant for $\mathrm{Rep}_{h-2}\, A_1^{(1)}$ if and only if it is a modular invariant for $\mathcal{C}$. Indeed, as $\mathcal{T}$ is diagonal its commutant subspace depends only upon equalities amongst its entries. The $(-1)^{a-1}$ factor may be though of as adding a -1 factor in front of every other diagonal entry in $\mathcal{T}$. As, for $a_1, a_2 \in \mathcal{I}$, $\exp\left(\pi i\frac{a_1^2}{2h}\right) = \pm\exp\left(\pi i\frac{a_2^2}{2h}\right)$ implies that a_1 and a_2 share the same parity these extra factors do not affect the commutant subspace. Furthermore the $(-1)^{a+b}$ factor appearing in $\mathcal{S}_{ab}$ may be removed by conjugating $\mathcal{S}$ with $W_{ab} = (-1)^{a-1}\delta_{a,b}$. As commuting with $\mathcal{T}$ forces invariance under conjugation by W this implies the claim.

5.2 | Module Categories over TL

5.2.1 | A Recipe for Module Categories over TL

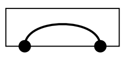
As described in Section 4.4.1 a finite module category $\mathcal{M}$ over TL is equivalent to a (monoidal) functor

$$\mathcal{M}\colon \mathrm{TL} \to \mathcal{D}$$

where $\mathcal{D} = A, A\text{-Bimod}$ and A is a finite direct sum of division algebras over $\mathbb{C}$ (see Remark 4.4.2). As $\mathbb{C}$ is algebraically closed, the only division algebra over $\mathbb{C}$ is $\mathbb{C}$ itself [Coh03, Proposition 5.4.5]. Therefore A is simply a direct sum of copies of $\mathbb{C}$, we write this decomposition as $A = \bigoplus_{i \in A_0} \mathbb{C}$.

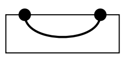
Let B denote the object in $\mathcal{D}$ such that $B = \mathcal{F}(\underline{1})$. As $\mathcal{M}$ is monoidal we have $\mathcal{M}(\underline{k}) = B^{\otimes k}$ and (as $\underline{1}$ is self dual) ${}_iB_j \cong {}_jB_i{}^*$. Furthermore the module category is fully described by B and the annihilation map

$$\mathcal{M}\Bigg(\quad\Bigg) : B \otimes B \to A$$



that we denote ϕ, and the creation map

$$\mathcal{M}\Bigg(\quad\Bigg) \colon A \to B \otimes B$$



that we denote φ.

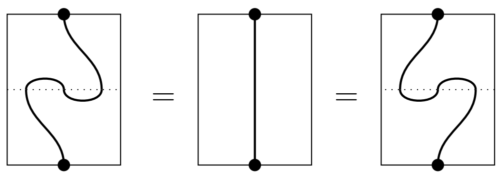
One way to specify such maps is to first choose a perfect pairing $\langle -, - \rangle$ between ${}_iB_j$ and ${}_jB_i$ then choose an invertible element $x = (x_i) \in A$ and set

$$\begin{aligned} \phi_{ij} \colon {}_iB_j \otimes {}_jB_i &\to \mathbb{C} \\ v \otimes w &\mapsto x_j \langle v, w \rangle \end{aligned} \tag{5.5}$$

and

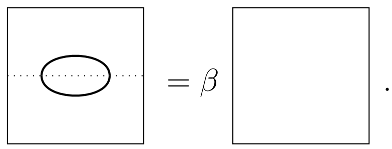
$$\begin{aligned} \varphi_{ij} \colon \mathbb{C} &\to {}_iB_j \otimes {}_jB_i \\ 1 &\mapsto \frac{1}{x_i} \sum_b b \otimes b^*. \end{aligned} \tag{5.6}$$

These two maps are subject to conditions arising from the following equations in TL,



(5.7)

and



(5.8)

A functor $\mathcal{M}$ given by (5.5) and (5.6) automatically satisfies (5.7) as, for $a \in {}_iB_j$, we have

$$\begin{aligned} (\phi_{ij} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \varphi_{ji})\,(a) &= (\phi_{ij} \otimes \mathrm{id}) \left(\frac{1}{x_j} a \otimes \sum_b b^* \otimes b \right) \\ &= \sum_b b^*(a) b = a \end{aligned}$$

and similarly $(\mathrm{id} \otimes \phi_{ji}) \circ (\varphi_{ij} \otimes \mathrm{id})\,(a) = a$. Furthermore $\mathcal{M}$ will satisfy (5.8) if and only if

$$\beta = \sum_{j \in A_0} \phi_{ij} \circ \varphi_{ij}(1) = \sum_{j \in A_0} \frac{x_j \dim {}_iB_j}{x_i}$$

for all $i \in A_0$. This holds if and only if β is an eigenvalue of B with eigenvector x. Furthermore (5.7) and (5.8) are the only relations in TL and we therefore have that (5.5) and (5.6) define a module category over TL if and only if β is an eigenvalue of ${}_iB_j$ with eigenvector x.

One advantage of considering module categories of this form is that they *induce a pivotal structure on the full image* (cf. Section 4.4.2). To prove this we first consider the following lemma.

LEMMA 5.2.1. *Let ϕ^n and φ^n denote the image of $\mathcal{M}(\mathrm{cr}_{\underline{n}})$ and $\mathcal{M}(\mathrm{an}_{\underline{n}})$ respectively. For $i, j \in Q_0$ and $n \in \mathbb{N}^+$ we have the following*

$$\phi^n_{ji}(w \otimes v) = \frac{x_i}{x_j}\, \phi^n_{ij}(v \otimes w) \tag{5.9}$$

and

$$\varphi^n_{ji} = \frac{x_i}{x_j}\, T^n_{ij} \circ \varphi^n_{ij} \tag{5.10}$$

where T^n_{ij} is the canonical isomorphism from ${}_iB^{\otimes n}{}_j \otimes {}_jB^{\otimes n}{}_i$ to ${}_jB^{\otimes n}{}_i \otimes {}_iB^{\otimes n}{}_j$.

Proof. We proceed by induction on n. The base case $n = 1$ is clear. Assuming the hypothesis for all integers up to $n-1$, we take $b \in {}_iB_k,\ v \in {}_kB^{\otimes n-1}{}_j,\ w \in {}_jB^{\otimes n-1}{}_k$ and compute,

$$\begin{aligned}\phi^n_{ij}(b \otimes v \otimes w \otimes b^*) &= \phi^{n-1}_{kj}(v \otimes w)\phi_{ik}(b \otimes b^*)\\ &= x_k \phi^{n-1}_{kj}(v \otimes w).\end{aligned}$$

We then also have

$$\begin{aligned}\phi^n_{ji}(w \otimes b^* \otimes b \otimes v) &= \phi^{n-1}_{jk}(w \otimes v)\phi_{ki}(b^* \otimes b)\\ &= x_i \phi^{n-1}_{jk}(w \otimes v)\\ &= \frac{x_i x_k}{xj}\phi^{n-1}_{kj}(v \otimes w)\\ &= \frac{x_i}{xj}\phi^n_{ij}(b \otimes v \otimes w \otimes b^*).\end{aligned}$$

Therefore (5.9) is proved. To prove (5.10) we proceed more directly,

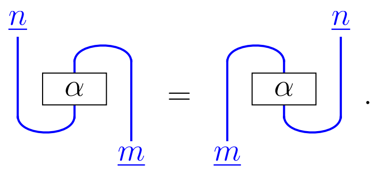
$$
\begin{aligned}
\frac{x_i}{x_j} T^n_{ij} \circ \varphi^n_{ij} &= \frac{x_i}{x_j} T^n_{ij} \circ \left(\sum_k (\mathrm{id}_{\underline{n-1}} \otimes \varphi_{kj} \otimes \mathrm{id}_{\underline{n-1}}) \circ \varphi^{n-1}_{ik} \right) \\
&= \frac{x_i}{x_j} \sum_k (\mathrm{id}_{\underline{1}} \otimes (T^{n-1}_{ik} \circ \varphi^{n-1}_{ik}) \otimes \mathrm{id}_{\underline{1}}) \circ (T^1_{kj} \circ \varphi_{kj}) \\
&= \sum_k (\mathrm{id}_{\underline{1}} \otimes \varphi^{n-1}_{ki} \otimes \mathrm{id}_{\underline{1}}) \circ \varphi_{jk} \\
&= \varphi^n_{ji}
\end{aligned}
$$

as desired.

□

PROPOSITION 5.2.2. *Let $\mathcal{M}$ be a module category on* TL *given by* (5.5) *and* (5.6). *Then* (4.17) *commutes. In other words, $\mathcal{M}$ induces a pivotal structure on its full image.*

Proof. As the pivotal structure on TL is given by the identity (4.17) reduces to ${}^\vee\alpha = \alpha^\vee$ for all $\alpha \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}(\underline{m}), \mathcal{M}(\underline{n}))$. Using the graphical notation of Chapter 4 the equation ${}^\vee\alpha = \alpha^\vee$ may be written as



For $a \in {}_iB^{\otimes n}{}_j$, by (5.9), we have

$$
\begin{aligned}
\alpha^\vee(a) &= (\mathrm{id} \otimes \phi^m_{ji}) \circ (\mathrm{id} \otimes \alpha_{ji} \otimes \mathrm{id}) \circ (\varphi^n_{ij} \otimes \mathrm{id})(a) \\
&= \sum_{IJ} \lambda^n_{IJ} \phi^m_{ji}(\alpha_{ji}(b_J) \otimes a)\, b_I \\
&= \frac{x_i}{x_j} \sum_{IJ} \lambda^n_{IJ} \phi^m_{ij}(a \otimes \alpha_{ji}(b_J))\, b_I
\end{aligned}
$$

where the λ^n_{IJ}, the b_I and the b_J are such that

$$
\varphi^n_{ij}(1) = \sum_{IJ} \lambda^n_{IJ}\, b_I \otimes b_J \in {}_iB^{\otimes n}{}_j \otimes {}_jB^{\otimes n}{}_i.
$$

However, by (5.10), we also have

$$\begin{aligned}{}^{\vee}\alpha(a) &= (\phi^m_{ij}\otimes \mathrm{id})\circ(\mathrm{id}\otimes\alpha_{ji}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\varphi^n_{ji})(a)\\ &= \frac{x_i}{x_j}\sum_{IJ}\lambda^n_{IJ}\phi^m_{ij}(a\otimes\alpha_{ji}(b_J))\, b_I\\ &= \alpha^{\vee}(a)\end{aligned}$$

and so we are done.

□

5.2.2 | Module Categories over TL and Symmetric Quivers

Let q be a root of unity with Coxeter number h and let $\beta = -[2]_q$. We recall that $\mathcal{C}$ denotes the idempotent completion of the quotient of $\mathrm{TL}(\beta)$ by the tensor ideal generated by e_{h-1}. In this section we find it useful to recall that, when $A = \bigoplus_{i\in A_0}\mathbb{C}$, an object in $A, A\text{-Bimod}$ is equivalent to a quiver $\mathcal{Q}$ with vertices $\mathcal{Q}_0 = A_0$.

To construct a module category $\mathcal{M}\colon \mathrm{TL} \to A, A\text{-Bimod}$ in the manner described in Section 5.2.1 we have to find a symmetric quiver $\mathcal{Q}$ with non-degenerate[†] eigenvalue β. However as our real goal is to find module categories over $\mathcal{C}$, the adjacency matrix of $\mathcal{Q}$ should also be bounded with respect to the A_1 recurrence relation

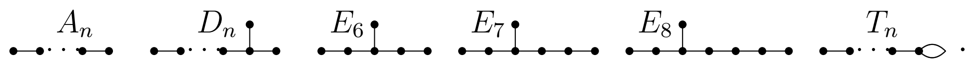
$$\begin{aligned}D_0 &= \mathrm{id},\\ D_1 &= \mathrm{adj}\, Q\\ D_{k+1} &= X_k(\mathrm{adj}\,\mathcal{Q}) - X_{k-1} \quad k\geq 1.\end{aligned}$$

Indeed, from the fusion rules given by Theorem 5.1.4 we see that D_k will be the dimension matrix of the image object associated to the idempotent $\mathcal{M}(e_k)$ in $A, A\text{-Bimod}$. Therefore, for $\mathcal{M}$ to be well defined on $\mathcal{C}$ a necessary condition is $D_{h-1} = 0$. This forces the recurrence relation to have periodicity $2(h-1)$

[†] A *non-degenerate* eigenvalue is an eigenvalue with non-zero entries.

which in turn forces it to be bounded. A quiver is bounded with respect to the A_1 recurrence relation if and only if its maximal eigenvalue is strictly less than 2 [Coo07, Section 3.3.2].

The list of connected symmetric quivers with maximal eigenvalue strictly less that 2 is given by the *double Dynkin quivers of type A, D, E and T* (see, for example [GdlHJ89]).



The characteristic polynomial of the A_n quiver is the *ultraspherical polynomial* P_n, defined recursively by

$$\begin{aligned}
&P_0(x) = 1,\\
&P_1(x) = x,\\
&P_{n+1}(x) = xP_n(x) - P_{n-1}(x) \quad n \geq 1.
\end{aligned}$$

The set of roots of P_n, and therefore the spectrum of the A_n quiver, is given by $\{-[2]_q \mid q = e^{\frac{\pi i l}{n+1}}, 1 \leq l \leq n\}$. Furthermore the spectrum of every double Dynkin quiver of type A, D, E or T is a subset of the set of roots of P_{h-1} for a certain value of h called the *Coxeter number of the quiver* [GdlHJ89]. For a double Dynkin quiver of type A, D, E or T to have non-degenerate eigenvalue $-[2]_q$ it is necessary that the diagram share its Coxeter number with q.

When h is an odd positive integer there are two possible values of q with Coxeter number h: q may be a primitive $2h$ root of unity (we call q *even*) or a primitive h root of unity (we call q *odd*). Table 5.2.2 describes when $-[2]_q$ is a non-degenerate eigenvalue of a double Dynkin quiver of type A, D, E or T (we note that if q has an even Coxeter number it is necessarily even itself).

Furthermore, whenever a double Dynkin quiver of type A, D, E or T has non-degenerate eigenvalue $-[2]_q$ the resulting module category $\mathcal{M}\colon \mathrm{TL} \to A, A\text{-Bimod}$ maps $e_{h-1} \in \mathrm{End}_{\mathrm{TL}}(\underline{h-1})$ to 0 and gives a well defined module category over $\mathcal{C}$ [Coo07, Proposition 5.6.10]. Further still, we have the following result from Etingof and Ostrik.

Type	h	q even	q odd
A_{2n}	$2n+1$	Yes	Yes
A_{2n+1}	$2n+2$	Yes	-
D_n	$2n-2$	Yes	-
E_6	12	Yes	-
E_7	18	Yes	-
E_8	30	Yes	-
T_n	$2n+1$	No	Yes

Table 5.1: When $-[2]_q$ is a non-degenerate eigenvalue of the A-D-E-T double Dynkin quivers.

THEOREM 5.2.3 ([EO04], Theorem 3.12). *Indecomposable module categories over $\mathcal{C}$ are classified by the double Dynkin quivers of type A, D, E and T that have $-[2]_q$ as a non-degenerate eigenvalue.*

COROLLARY 5.2.4. *Every module category over $\mathcal{C}$ induces a pivotal structure on its full image.*

Proof. This follows immediately from Theorem 5.2.3 and Proposition 5.2.2. □

For $\mathcal{C}$ a PTC and $\mathcal{M}\colon \mathcal{C} \to A, A\text{-Bimod}$ an arbitrary module category the complexified Grothendieck ring of $\mathcal{C}$, denoted $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$, has a natural action on the complexified Grothendieck ring of $\text{Mod-}A$ given by

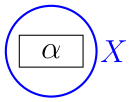
$$[X] \cdot [V] = [\mathcal{M}(X) \otimes_A V].$$

Furthermore, as $A = \bigoplus_{i \in A_0} \mathbb{C}$, indecomposable objects in $\text{Mod-}A$ are indexed by A_0. In particular, for $j \in \mathcal{A}_0$ we have

$$[X] \cdot [V_j] = \sum_i \dim {}_i\mathcal{M}(X)_j \, [V_i]$$

where V_j is the indecomposable module corresponding to j.

Let $\mathcal{C}$ be as defined in Section 5.1.4 and let $\mathcal{M}\colon \mathcal{C} \to A, A\text{-Bimod}$ be a module category. By Theorem 5.2.3 there exists an invertible element $x \in A$ such that $\mathcal{M}$ is given by (5.5) and (5.6) with respect to this x. By Corollary 5.2.4 we may consider $\mathcal{TM}$ as constructed in Section 4.1. Recall from Remark 3.1.5 that $\mathrm{End}_{\mathcal{TC}}(\mathbf{1}) = \mathcal{K}_{\mathbb{C}}(\mathcal{C})$. Therefore, as $\mathcal{TM}$ is a functor, this defines an action of $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$ on $\mathrm{End}_{\mathcal{D}}(A)$ given by

$[X] \cdot \alpha =$ 

.

PROPOSITION 5.2.5. *The map*

$$\begin{aligned} \Phi\colon \mathrm{End}_{\mathcal{D}}(A) &\to \mathcal{K}_{\mathbb{C}}(\text{Mod-}A) \\ \mathbf{1}_j &\mapsto x_j[V_j] \end{aligned}$$

is an isomorphism of $\mathcal{K}_{\mathbb{C}}(\mathcal{C})$-modules.

Proof. As $\{[V_j]\}$ and $\{\mathbf{1}_j\}$ form a basis of $\mathcal{K}_{\mathbb{C}}(\text{Mod-}A)$ and $\mathrm{End}_{\mathcal{D}}(A)$ respectively Φ is an isomorphism of vector spaces. As, for X in $\mathcal{C}$,

$$[X] \cdot \mathbf{1}_j = \sum_i \phi_{ij}^X \circ \varphi_{ij}^X \, \mathbf{1}_i = \sum_i \frac{x_j}{x_i} \dim {}_i\mathcal{M}(X)_j \, \mathbf{1}_i$$

where ϕ_{ij}^X and φ_{ij}^X is $\mathcal{M}(\mathrm{cr}_X)$ and $\mathcal{M}(\mathrm{an}_X)$ respectively, we have

$$\begin{tikzcd}
\mathbf{1}_j \arrow[r, mapsto, "{[X]}"] \arrow[d, mapsto, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \sum_i \frac{x_j}{x_i} \dim {}_i\mathcal{M}(X)_j \, \mathbf{1}_i \\
x_j[V_j] \arrow[r, mapsto, "{[X]}"] & x_j \sum_i \dim {}_i\mathcal{M}(X)_j \, [V_i] \arrow[u, mapsto, "\Phi^{-1}"']
\end{tikzcd}$$

as desired.

∎

5.2.3 | The Cappelli-Itzykson-Zuber Classification

Let q be a primitive even root of unity with Coxeter number h and let $\mathcal{C}$ be the corresponding MTC described in Section 5.1.4.

In mathematical physics a *Wess-Zumino-Witten (WZW) model* is a type of two-dimensional conformal field theory, the starting ingredients of which are an affine Lie algebra $\mathfrak{g}$ and a level $k \in \mathbb{Z}^+$. The modular tensor category associated to a WZW model is then the category of integrable, highest weight modules of $\mathfrak{g}$ at level k, denoted $\operatorname{Rep}_k \mathfrak{g}$. We recall from Section 5.1.4 that when $\mathfrak{g} = A_1^{(1)}$, $\operatorname{Rep}_{h-2} \mathfrak{g}$ is equivalent (up to a difference in pivotal structure) to $\mathcal{C}$.

In 1986 Cappelli, Itzykson and Zuber [CIZ87] classified all modular invariant partition functions for the WZW model associated to the $\mathfrak{g} = A_1^{(1)}$ case. In other words they gave a complete list of modular invariants, i.e. non-negative integer matrices Z that commute with the modular data associated to $\operatorname{Rep}_{h-2} \mathfrak{g}$ and satisfy $Z_{11} = 1$ (See Section 1.1).

Theorem 5.2.6 (C.I.Z. Classification). *The complete list of modular invariants associated to $A_1^{(1)}$ at level $h-2$ is as follows. To aid legibility, we present these modular invariants as partition functions, cf.* (1.2).

$$
\begin{aligned}
\mathcal{A}_{h-1} &= \sum_{a=1}^{h-1} |\chi_a|^2 , && \forall h \geq 3\\
\mathcal{D}_{\frac{h}{2}+1} &= \sum_{a=1}^{h-1} \chi_a \, \chi^*_{J^{a-1}a} , && \text{whenever } \frac{h}{2} \text{ is even}\\
\mathcal{D}_{\frac{h}{2}+1} &= |\chi_1 + \chi_{J1}|^2 + |\chi_3 + \chi_{J3}|^2 + \cdots + 2|\chi_{\frac{h}{2}}|^2 , && \text{whenever } \frac{h}{2} \text{ is odd}\\
\mathcal{E}_6 &= |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 , && \text{for } h = 12\\
\mathcal{E}_7 &= |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 \\
&\quad + \chi_9 \left(\chi_3 + \chi_{15}\right)^* + \left(\chi_3 + \chi_{15}\right) \chi_9^* + |\chi_9|^2 , && \text{for } h = 18\\
\mathcal{E}_8 &= |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 , && \text{for } h = 30.
\end{aligned}
$$

where $J : \{1, 2, ..., h-1\} \to \{1, 2, ..., h-1\}$ maps a to $h-a$.

An important feature of this classification is that it obeys the following A-D-E pattern. Let $\mathcal{X}$ be a double Dynkin quiver of type A,D or E. Recall from the previous section that eigenvalues of $\mathcal{X}$ form a subset of $\{-[2]_q \mid q = e^{\frac{\pi i l}{h}}, 1 \leq l \leq h-1\}$ where h is the Coxeter number of the quiver. Then, for $l \in \{1, 2, \ldots h-1\}$, the lth diagonal entry in the modular invariant associated to $\mathcal{X}$ gives the dimension of the corresponding eigenspace of $\mathcal{X}$. The story of how this A-D-E pattern was first noticed is an entertaining anecdote documented by Gannon in [Gan00b],

«Around Christmas 1985, Zuber wrote Kac about the $A_1^{(1)}$ physical invariant problem, and mentioned the physical invariants he and Itzykson knew at that point (what we now call $\mathcal{A}_\bullet$ and $\mathcal{D}_{\text{even}}$). A few weeks later, Kac wrote back saying he found one more invariant, and jokingly pointed out that it must be indeed quite exceptional as the exponents of $\mathcal{E}_6$ appeared in it. "I must confess that I didn't pay much attention to that last remark (I hardly knew what Coxeter exponents were, at the time!)". By spring 1986, Cappelli arrived in Paris and got things moving again; together Cappelli-Itzykson-Zuber found $\mathcal{E}_7, \mathcal{D}_{\text{odd}}$ and then $\mathcal{E}_8$, and struggled to find more. "And it is only in August, during a conversation with Pasquier, in which he was showing me his construction of lattice models based on Dynkin diagrams, that I suddenly remembered this cryptic but crucial observation of Victor, rushed to the library to find a list of the exponents of the other algebras... and found with the delight that you can imagine that they were matching our list". Thus the A-D-E pattern to these physical invariants was discovered.».

This pattern becomes all the more intriguing in light of the classification result given by Theorem 5.2.3. Indeed, the theorem states that indecomposable module categories over $\mathcal{C}$ are classified by the double Dynkin quivers of type A,D and E (the T type does not appear as q is a primitive *even* root of unity cf. Table 5.2.2). Obtaining a greater understanding of the relationship between these two classification results was one of the motivating goals for the work in this thesis.

The $\mathcal{TM}$ construction explains the A-D-E pattern appearing in the classification of $A_1^{(1)}$ modular invariants in the following way. Let $\mathcal{X}$ be an A-D-E double Dynkin quiver and let $\mathcal{M}\colon \mathcal{C} \to A, A\text{-Bimod}$ be the corresponding module category over $\mathcal{C}$. It is known that applying α-induction, as described in

Section 4.4.1, to $\mathcal{M}$ yields the modular invariant associated to $\mathcal{X}$ by the list appearing in Theorem 5.2.6 [BE01, Section 5]. We denote this modular invariant Z. By Theorem 4.4.6 the entries of Z may be thought of as the dimensions of the simple multiplicity spaces in $\mathcal{TM}$, in other words

$$Z = Z(\mathcal{TM})$$

where $Z(\mathcal{TM})$ is given by Definition 4.1.3. We recall that $\mathrm{End}_{\mathcal{TC}}(\mathbf{1})$ is a semisimple commutative algebra generated by the orthogonal primitive idempotents $\{\mathbf{1}_I\}_{I\in\mathrm{Irr}(\mathcal{C})}$ where $(\mathbf{1}, \mathbf{1}_I)^\sharp = (II^\vee, e_I^{I\vee})^\sharp$, see Section 3.1.3. The diagonal terms in Z therefore correspond to the dimensions of the weight spaces of the action of $\mathrm{End}_{\mathcal{TC}}(\mathbf{1}) = \mathcal{K}_\mathbb{C}(\mathcal{C})$ on $TM(\mathbf{1})$. However by Proposition 5.2.5 this action coincides with the action of $\mathcal{K}_\mathbb{C}(\mathcal{C})$ on $\mathcal{K}_\mathbb{C}(\mathrm{Mod}\text{-}A)$ described in Section 5.2.2. As the weight spaces of this action are given by the eigenspaces of $\mathcal{X}$ this explains the pattern.

CHAPTER 6

CONCLUSION

6.1 | REFLECTIONS

When reflecting upon the results in this thesis it is interesting to note that, in an attempt to better understand the solution to a relatively low-tech problem, namely the search for modular invariants i.e. non-negative integer solutions in a commutant space, we are lead to a higher categorical result, namely that $\mathcal{TM}$ is a Frobenius algebra. Indeed, other research has previously suggested that modular invariants are perhaps simply best thought of as algebra objects in the centre of the category (see, for example, [KR09, FRS02]). This ties nicely into the wider research theme of exploiting higher category theory to rephrase and extend pre-existing methods in algebraic conformal field theory.

There are pre-existing methods for relating module categories to modular invariant† Frobenius algebras in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. Any module category over $\mathcal{C}$ may be realised (non-uniquely) as the category of modules of an algebra in $\mathcal{C}$ [Ost03]. The *full centre construction* [FFRS08, Definition 4.9] then associates a modular invariant, commutative, symmetric Frobenius algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ to a special (as defined in, for example, [KR09]), symmetric Frobenius algebra in $\mathcal{C}$ [KR09, Theorem 3.18]. Furthermore every modular invariant, commutative, symmetric Frobenius algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ may be realised in this way [KR09, Theorem 3.22].

† As defined in [Kon08, Section 6], cf. Remark 4.3.5.

The full centre construction may also be described in terms of the module category directly [DKR15, Section 3.1]. Schaumann has worked on characterising the condition that the module category be equivalent to the category of modules of a special symmetric Frobenius algebra purely in terms of the module category itself. In particular he has shown that it is equivalent to requiring that the module category admits a *module trace* [Sch13]. It is possible that this could be related to the condition identified in this thesis: that the module category induce a pivotal structure on its full image.

Another moral one may extract from this work is to never underestimate the pivotal structure. Indeed there are many examples in the literature of the pivotal structure being underestimated. There is also a recurring theme of researchers finally clearing up confusing details via careful consideration of the pivotal structure (see, for example, [Tin17]). The appearance of the condition that a functor induce a pivotal structure on its full image was unexpected. This condition is surprising in its subtlety and it would be interesting to know if it may be rephrased in some way.

6.2 | Future Work

There are several possible ways the work presented in this thesis could be extended. Perhaps the most obvious would be to search for more examples of module categories that induce pivotal structures on their full images. For example, I strongly expect that this is the case for the $A_2^{(1)}$ modules categories given by the Di Francesco-Zuber quivers [DFZ90]. Furthermore, it is known† that these quivers classify $A_2^{(1)}$ module categories in an analogous manner to the classification of $A_1^{(1)}$ modules categories seen in Section 5.2.2.

The recipe for constructing a module category from a Di Francesco-Zuber quiver is an extension of the the recipe described in Section 5.2.1 [Coo07, Section 7.3]. However, as $\operatorname{Rep}_k A_2^{(1)}$ is tensor generated by two dual objects (as opposed to TL which is tensor generated by a single self-dual object) the quiver will not necessarily be symmetric and its dual will also play a role. As in the $A_1^{(1)}$

†In personal communication Professor David E. Evans has informed me that this classification is known to experts but does not yet appear in the literature.

case, the Di Francesco-Zuber quivers admit the relevant quantum dimension as a non-degenerate eigenvalue with the additional property that they share the corresponding eigenvector with their duals. This enables us to define the functor on the dualising morphisms in $\mathrm{Rep}_k A_2^{(1)}$ in a manner analogous to the approach taken in Section 5.2.1. I therefore expect that the $\mathrm{Rep}_k A_2^{(1)}$ analogues of Corollary 5.2.4 and Proposition 5.2.5 should hold. Furthermore, the list of Di Francesco-Zuber quivers and of indecomposable module categories over $\mathrm{Rep}_k A_2^{(1)}$ both follow a (more complex) A-D-E pattern.

It would be valuable to find interesting examples of monoidal functors that *fail* to induce pivotal structures on their full images (Example 4.4.5 feels fairly pathological). In particular it would be interesting to known if there exists a monoidal functor $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$ that fails to induce a pivotal structure on its full image *for all choices of pivotal structure* on $\mathcal{C}$ (Example 4.4.5 does not satisfy this property).

The principal claim of α-induction (see Section 4.4) suggests the following conjecture.

CONJECTURE 6.2.1. *Let $\mathcal{C}$ be an MTC over an algebraically closed field and let $\mathcal{M}$ be a indecomposable module category over $\mathcal{C}$ that induces a pivotal structure on its full image. Then $Z(\mathcal{T}\mathcal{M})$ is a modular invariant.*

As discussed at the start of Section 4.3, Theorem 4.4.8 reduces a proof of this conjecture to a proof that $d(\mathcal{T}\mathcal{M}) = d(\mathcal{C})$. However, the graphical approach taken throughout this thesis seems ill-suited to tackling this problem. In particular it is unclear how, for an arbitrary pivotal monoidal functor $\mathcal{M}\colon \mathcal{C} \to \mathcal{D}$, the graphical calculus could distinguish the case when $\mathcal{D} = A, A\text{-Bimod}$ for some semisimple algebra A, i.e. the case when $\mathcal{M}$ is a module category.

One possible way of extending the $\mathcal{T}\mathcal{M}$ construction more generally would be to work on removing the assumption that $\mathcal{C}$ is a *rational* MTC, i.e. with finitely many isomorphism classes of simple objects. One could start by extending the main results to logarithmic conformal field theories as their connection to the rational case is well understood from a categorical point of view (see, for example, the recent work of Gannon et al. in particular [EG17], [CG17] and [FGSS18]).

«Boys and girls in cars
Dogs and birds on lawns
From here I can touch the sun

Put your jackets on
I feel we're being born
The tropic of Capricorn is below

We stall above the pole
Still your face is young
As we feel our weight return

A trail of shooting stars
The horses call the storm
Because the air contains the charge

The radio is on
And Houston knows the score
Can you feel it we're almost home

The crew compartment's breaking up
The crew compartment's breaking up
The crew compartment's breaking up
This is all I wanted to bring home to you
The crew compartment's breaking up
This is all I wanted to bring home to you
The crew compartment's breaking up
This is all I wanted to bring home to you »

—*The Commander Thinks Aloud* by The Long Winters

BIBLIOGRAPHY

[BE98] J. Böckenhauer and D. E. Evans. Modular invariants, graphs and α-induction for nets of subfactors. I. *Comm. Math. Phys.*, 197(2):361–386, 1998.

[BE01] Jens Böckenhauer and David E. Evans. Modular invariants and subfactors. In *Mathematical physics in mathematics and physics (Siena, 2000)*, volume 30 of *Fields Inst. Commun.*, pages 11–37. Amer. Math. Soc., Providence, RI, 2001.

[BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[Car89] John L. Cardy. Boundary conditions in conformal field theory. In *Integrable systems in quantum field theory and statistical mechanics*, volume 19 of *Adv. Stud. Pure Math.*, pages 127–148. Academic Press, Boston, MA, 1989.

[CG17] Thomas Creutzig and Terry Gannon. Logarithmic conformal field theory, log-modular tensor categories and modular forms. *J. Phys. A*, 50(40):404004, 37, 2017.

[CIZ87] A. Cappelli, C. Itzykson, and J.-B. Zuber. The A-D-E classification of minimal and $A_1^{(1)}$ conformal invariant theories. *Comm. Math. Phys.*, 113(1):1–26, 1987.

[CMW09] David Clark, Scott Morrison, and Kevin Walker. Fixing the functoriality of Khovanov homology. *Geom. Topol.*, 13(3):1499–1582, 2009.

[Coh03] P. M. Cohn. *Basic algebra*. Springer-Verlag London, Ltd., London, 2003. Groups, rings and fields.

[Coo07] Barrie Cooper. *Almost Koszul Duality and Rational Conformal Field Theory*. PhD thesis, University of Bath, 2007.

[DFZ90] P. Di Francesco and J.-B. Zuber. $\mathrm{SU}(N)$ lattice integrable models associated with graphs. *Nuclear Phys. B*, 338(3):602–646, 1990.

[DKR15] Alexei Davydov, Liang Kong, and Ingo Runkel. Functoriality of the center of an algebra. *Adv. Math.*, 285:811–876, 2015.

[EG17] David E. Evans and Terry Gannon. Non-unitary fusion categories and their doubles via endomorphisms. *Adv. Math.*, 310:1–43, 2017.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.

[EO04] Pavel Etingof and Viktor Ostrik. Module categories over representations of $\mathrm{SL}_q(2)$ and graphs. *Math. Res. Lett.*, 11(1):103–114, 2004.

[FFRS08] Jens Fjelstad, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. Uniqueness of open/closed rational CFT with given algebra of open states. *Adv. Theor. Math. Phys.*, 12(6):1283–1375, 2008.

[FGSS18] Jürgen Fuchs, Terry Gannon, Gregor Schaumann, and Christoph Schweigert. The logarithmic Cardy case: boundary states and annuli. *Nuclear Phys. B*, 930:287–327, 2018.

[FRS02] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. TFT construction of RCFT correlators. I. Partition functions. *Nuclear Phys. B*, 646(3):353–497, 2002.

[Gan00a] Terry Gannon. The Cappelli-Itzykson-Zuber A-D-E classification. *Rev. Math. Phys.*, 12(5):739–748, 2000.

[Gan00b] Terry Gannon. Monstrous Moonshine and the classification of CFT. In *Conformal field theory (Istanbul, 1998)*, volume 102 of *Front. Phys.*, page 66. Adv. Book Program, Perseus Publ., Cambridge, MA, 2000.

[GdlHJ89] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989.

[GW02] F. M. Goodman and H. Wenzl. Ideals in the temperley lieb category. arXiv:math/0206301v1, 2002.

[HK19] Leonard Hardiman and Alastair King. Decomposing the tube category. *Glasgow Mathematical Journal*, 2019.

[Hua97] Yi-Zhi Huang. *Two-dimensional conformal geometry and vertex operator algebras*, volume 148 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1997.

[Hua05] Yi-Zhi Huang. Vertex operator algebras, the Verlinde conjecture, and modular tensor categories. *Proc. Natl. Acad. Sci. USA*, 102(15):5352–5356, 2005.

[Jr11] Alexander Kirillov Jr. String-net model of turaev-viro invariants. arXiv:1106.6033v1, 2011.

[Kas98] Ch. Kassel. Quantum groups. In *Algebra and operator theory (Tashkent, 1997)*, pages 213–236. Kluwer Acad. Publ., Dordrecht, 1998.

[KO02] Alexander Kirillov, Jr. and Viktor Ostrik. On a q-analogue of the McKay correspondence and the ADE classification of $\mathfrak{sl}_2$ conformal field theories. *Adv. Math.*, 171(2):183–227, 2002.

[Kon08] Liang Kong. Cardy condition for open-closed field algebras. *Comm. Math. Phys.*, 283(1):25–92, 2008.

[KR09] Liang Kong and Ingo Runkel. Cardy algebras and sewing constraints. I. *Comm. Math. Phys.*, 292(3):871–912, 2009.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Müg03] Michael Müger. On the structure of modular categories. *Proc. London Math. Soc. (3)*, 87(2):291–308, 2003.

[Ocn94] Adrian Ocneanu. Chirality for operator algebras. In *Subfactors (Kyuzeso, 1993)*, pages 39–63. World Sci. Publ., River Edge, NJ, 1994.

[Ocn99] Adrian Ocneanu. Paths on coxeter diagrams: From platonic solids and singularities to minimal models and subfactors. In *Lectures on operator theory*, volume 13 of *Fields Inst. Monographs*. Amer. Math. Soc., Providence, 1999. Notes by S. Goto.

[Ost03] Victor Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8(2):177–206, 2003.

[Pen71] Roger Penrose. Applications of negative dimensional tensors. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 221–244. Academic Press, London, 1971.

[PSV18] Sorin Popa, Dimitri Shlyakhtenko, and Stefaan Vaes. Cohomology and L^2-Betti numbers for subfactors and quasi-regular inclusions. *Int. Math. Res. Not. IMRN*, (8):2241–2331, 2018.

[RT90] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.*, 127(1):1–26, 1990.

[Sch13] Gregor Schaumann. Traces on module categories over fusion categories. *J. Algebra*, 379:382–425, 2013.

[Seg88] G. B. Segal. The definition of conformal field theory. In *Differential geometrical methods in theoretical physics (Como, 1987)*, volume 250 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 165–171. Kluwer Acad. Publ., Dordrecht, 1988.

[ST09] Noah Snyder and Peter Tingley. The half-twist for $U_q(\mathfrak{g})$ representations. *Algebra Number Theory*, 3(7):809–834, 2009.

[Sug68] Hirotaka Sugawara. A field theory of currents. *Phys. Rev.*, 170:1659–1662, Jun 1968.

[Tin17] Peter Tingley. A minus sign that used to annoy me but now I know why it is there (two constructions of the Jones polynomial). In *Proceedings of the 2014 Maui and 2015 Qinhuangdao conferences in honour of Vaughan F. R. Jones' 60th birthday*, volume 46 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 415–427. Austral. Nat. Univ., Canberra, 2017.

[Tur16] Vladimir G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2016. Third edition [of MR1292673].

[Wen87] Hans Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.

[Zub02] J.-B. Zuber. CFT, BCFT, ADE and all that. In *Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, volume 294 of *Contemp. Math.*, pages 233–266. Amer. Math. Soc., Providence, RI, 2002.